

BURGESS-LIKE SUBCONVEX BOUNDS FOR $GL_2 \times GL_1$

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ABSTRACT. Based on the method in [26], we give a Burgess-like subconvex bound, similar to that of [6], for $L(s, \pi \otimes \chi)$ in terms of the analytical conductor of χ , where π is a GL_2 cuspidal representation and χ is a Hecke character.

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1. INTRODUCTION

1.1. Statement of the Main Results. Let \mathbb{A} be the adèle ring of a number field F . Let π, π_1, π_2 be generic automorphic representations of $G(\mathbb{A}) = GL_2(\mathbb{A})$, where

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at least one of π_1, π_2 is cuspidal. Let χ be a Hecke character. Denote by $C(\pi)$ (resp. $C(\chi)$) the analytic conductor of π (resp. χ).

P. Michel and A. Venkatesh in [26] solved the subconvexity problem for GL_2 . In fact, the main result of that paper is the existence of some $\delta > 0$ such that

$$L(1/2, \pi_1 \times \pi_2) \ll_{F, \epsilon, \pi_1} C(\pi_2)^{1/4 - \delta + \epsilon}, \forall \epsilon > 0.$$

That is to say, if one fixes π_1 , then we have subconvex bound for $L(1/2, \pi_1 \otimes \pi_2)$ as $C(\pi_2)$ tends to infinity. As a preliminary result, they also obtained the following subconvex bound

$$L(1/2, \pi \times \chi) \ll_{F, \epsilon, \pi} C(\chi)^{1/2 - \delta + \epsilon}, \forall \epsilon > 0.$$

The main result of this paper is to give an explicit value of δ .

Theorem 1.1. *Let θ be such that no complementary series with parameter $> \theta$ appear as a component of a cuspidal automorphic representation of $G(\mathbb{A})$. For any cuspidal automorphic representation π of $G(\mathbb{A})$ and any Hecke character χ of analytic conductor $C(\chi) = Q$, we have*

$$L(1/2, \pi \otimes \chi) \ll_{F, \epsilon, \pi} Q^{1/2 - \delta + \epsilon}, \forall \epsilon > 0$$

with

$$\delta = \frac{1 - 2\theta}{8}.$$

Note that under the Ramanujan-Petersson conjecture ($\theta = 0$), $\delta = 1/8$.

Remark 1.2. *This bound, when $\theta = 0$, is called a Burgess bound. Burgess first obtained such bounds for Dirichlet characters for the level aspect in [3]. The best known value $\theta = 7/64$ is due to Kim and Sarnak in [23] over \mathbb{Q} , and to Blomer and Brumley in [4] over an arbitrary number field.*

Remark 1.3. *In [7], Blomer, Harcos and Michel first established such a Burgess-like bound in the level aspect for $F = \mathbb{Q}$. It was then generalized in [6] by Blomer and Harcos for any totally real number field F . The best bound for $F = \mathbb{Q}$, $\delta = 1/8$, in the level aspect is Theorem 2 of [5] by Blomer and Harcos. In the case $F = \mathbb{Q}$ and χ is quadratic, $\delta = 1/6$ was obtained by Conrey and Iwaniec as Corollary 1.2 of [14].*

1.2. Plan of the Paper. Section 2 is concerned with some technical but fundamental aspects of the proof of Theorem 1.1:

In section 2.1 we provide notations and conventions. In sections 2.2 to 2.4 we recall how Hecke's theory can be extended from K -finite vectors to smooth vectors. In section 2.5 we discuss Whittaker models and their norms. In sections 2.6 and 2.7, we discuss various forms of the spectral decomposition of automorphic functions. In section 2.8 we use results from the section 2.5 to construct and study local test vectors to be used in the sequel. In section 2.9 we discuss the decay of matrix coefficients of automorphic representations.

In section 3 we start the proof of Theorem 1.1, setting up the amplification method. We split to two sorts of arguments: local ones and global ones. The intuition behind the formal calculations is explained in Remark 3.11. It seems that the idea of translation by $n(T)$ originates from P. Sarnak in [28]. The whole idea is the combination of his idea together with the amplification method.

In section 4 we deal with the local arguments and prove Proposition 3.1. In section 5 we conclude the proof by putting local estimations into the global arguments.

The reader is strongly recommended to read Remark 3.11 before entering into the subsequent calculations. The difference in methods between this paper and [26] is explained in Remark 3.12.

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2. SOME PRELIMINARIES

2.1. Notations and Conventions. From now on, F is a number field of degree $r = [F : \mathbb{Q}] = r_1 + 2r_2$, where r_1 is the number of real places and r_2 is the number of complex places. V_F is the set of all places of F . For any $v \in V_F$, F_v is the completion of F at the place v . $\mathbb{A} = \mathbb{A}_F$ is the adèle ring of F . \mathbb{A}^\times is the idele group. We fix once for all an isometric section $\mathbb{R}_+ \rightarrow \mathbb{A}^\times$ of the adelic norm map $|\cdot| : \mathbb{A}^\times \rightarrow \mathbb{R}_+$, thus identify \mathbb{A}^\times with $\mathbb{R}_+ \times \mathbb{A}^{(1)}$ where $\mathbb{A}^{(1)}$ is the kernel of the adelic norm map. We'll constantly identify \mathbb{R}_+ with its image under the section map. Let $F_\infty = \prod_{v|\infty} F_v$ and $F_\infty^{(1)}$ be the subgroup of F_∞^\times of adelic norm 1. \mathbb{A}_f is the subring of finite adèles. \mathbb{A}_f^\times is the unit group of \mathbb{A}_f .

We denote by $\psi = \prod_v \psi_v$ the additive character $\psi = \psi_{\mathbb{Q}} \circ Tr_{F/\mathbb{Q}}$, where $\psi_{\mathbb{Q}}$ is the additive character of $\mathbb{Q} \backslash \mathbb{A}_{\mathbb{Q}}$ taking $e^{2\pi i x}$ on \mathbb{R} . At each place $v \in V_F$, dx_v denotes a self-dual measure w.r.t. ψ_v . Note if $v < \infty$, then dx_v is the measure which gives the ring of integers \mathcal{O}_v of F_v the measure $q_v^{-d_v/2}$, where q_v is the cardinal of the residue field of F_v , and $\prod_{v < \infty} q_v^{d_v}$ is the discriminant of F . We set $v(\psi) = -d_v$.

Define $dx = \prod_{v \in V_F} dx_v$ on \mathbb{A} . The quotient measure on $F \backslash \mathbb{A}$ has total volume 1 (c.f.

Proposition 7 [24] Chapter *XIV*). Define for $s \in \mathbb{C}$, if $v < \infty$ $\zeta_v(s) = (1 - q_v^{-s})^{-1}$, if v is real $\zeta_v(s) = \Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$, if v is complex $\zeta_v(s) = \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s)$.

Take $d^\times x_v = \zeta_v(1) \frac{dx_v}{|x_v|}$ as the Haar measure on the multiplicative group F_v^\times , and

$d^\times x = \prod_v d^\times x_v$ as the Haar measure on the idele group \mathbb{A}^\times .

Unless otherwise specified, $G = GL_2$ as an algebraic group defined over F . Hence $G_v = GL_2(F_v)$. If v is a complex place, then $K_v = SU_2(\mathbb{C})$; if v is a real place, then $K_v = SO_2(\mathbb{R})$; if $v < \infty$ then $K_v = G(\mathcal{O}_v)$. $Z_v = \left\{ z(u) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} : u \in F_v^\times \right\}$, $N_v = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F_v \right\}$, $A_v = \left\{ a(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} : y \in F_v^\times \right\}$. The probability Haar measure on K_v is dk_v . Z_v (resp. N_v resp. A_v) is equipped with the measure $d^\times u$ (resp. dx resp. $d^\times y$). Consider the Iwasawa decomposition $G_v = Z_v N_v A_v K_v$, a Haar measure of G_v is given by $dg_v = d^\times u dx d^\times y / |y|_v dk_v$, which in fact gives $K_v \subset G_v$ the measure $q_v^{-d_v}$ for $v < \infty$. View $Z_v \backslash G_v$ as $N_v A_v K_v$, equipped with the measure $d\bar{g}_v = dx d^\times y / |y|_v dk_v$. The center of $G(\mathbb{A})$ is

$Z = \prod_{v \in V_F} Z_v$. Denote $A = \prod_v A_v$. The quotient group $Z \backslash G(\mathbb{A})$ is equipped with the product measure $d\bar{g} = \prod_{v \in V_F} d\bar{g}_v$. The quotient measure on $X(F) = ZG(F) \backslash G(\mathbb{A})$ is also denoted by $d\bar{g}$, with total mass $\text{Vol}(X(F))$. $K = \prod_{v \in V_F} K_v$ is equipped with the product measure $dk = \prod_v dk_v$. Write $K_\infty = \prod_{v|\infty} K_v$ and $K_f = \prod_{v<\infty} K_v$.

Given a Hecke character ω , $L^2(G(F) \backslash G(\mathbb{A}), \omega)$ is the space of Borel functions φ satisfying

$$\forall \gamma \in G(F), \varphi(\gamma g) = \varphi(g); \forall z \in Z, \varphi(zg) = \omega(z)\varphi(g);$$

$$\|\varphi\|_{X(F)}^2 = \int_{X(F)} |\varphi(\bar{g})|^2 d\bar{g} < \infty.$$

Let $L_0^2(G(F) \backslash G(\mathbb{A}), \omega)$ be the (closed) subspace of cusp forms $\varphi \in L^2(G(F) \backslash G(\mathbb{A}), \omega)$, satisfying

$$\int_{F \backslash \mathbb{A}} \varphi(n(x)g) dx = 0, a.e. g \in G(\mathbb{A}).$$

Denote by R the right regular representation of $G(\mathbb{A})$ on $L^2(G(F) \backslash G(\mathbb{A}), \omega)$, and by R_0 the subrepresentation $L_0^2(G(F) \backslash G(\mathbb{A}), \omega)$. We know that each irreducible component π of R decomposes into $\pi = \hat{\otimes}_v' \pi_v$ where π_v are irreducible unitary representations of G_v . $R = R_0 \oplus R_{res} \oplus R_c$ is the spectral decomposition. R_0 decomposes as a direct sum of irreducible $G(\mathbb{A})$ -representations, whose components are called cuspidal representations. R_{res} is the sum of all one dimensional subrepresentations. R_c is a direct integral of irreducible $G(\mathbb{A})$ -representations, expressed via Eisenstein series. Components of R_0 and R_c are the generic automorphic representations. Let θ be such that no complementary series representation with parameter $> \theta$ appears as a local component of a cuspidal representation. Recall that a principal series representation $\pi(\mu_1, \mu_2) = \text{Ind}_{B(F_v)}^{G(F_v)}(\mu_1, \mu_2)$ with $|\mu_1 \mu_2^{-1}(t)| = |t|_v^s, \forall t \in F_v$ is a complementary series if s is a non-zero real number in the interval $(-1, 1)$. $|s|/2$ is called its parameter.

A compact open subgroup $K_f' \subset G(\mathbb{A}_f)$ is said to be of (congruence) type 0 if for every finite place v , there is an integer m_v such that the local component $K_v' = K_v^0[m_v] := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathcal{O}_v) \mid c \equiv 0 \pmod{\varpi_v^{m_v}} \right\}$, where ϖ_v is a uniformiser of the local field F_v . Let $\varphi \in \pi$ be a pure tensor vector in an automorphic representation. Suppose for every $v < \infty$, φ is invariant by $K_v^0[m_v]$ but not by $K_v^0[m_v - 1]$, then define $m_v = v(\varphi)$. Define $v(\pi) = v(\pi_v) = \min_{\varphi \in \pi_v} v(\varphi)$. The local conductor $C(\pi_v) = \varpi_v^{v(\pi_v)}$. We similarly define the principal congruence subgroups $K_v[n] := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathcal{O}_v) \mid a - 1, b, c, d - 1 \equiv 0 \pmod{\varpi_v^n} \right\}$.

For any semisimple (real) Lie group G , denote by \mathcal{C}_G the Casimir element. In our case, $G = GL_2$. At each place $v \mid \infty$, $Z_v \backslash G_v$ is semisimple, and $\Delta_v = -\mathcal{C}_{Z_v \backslash G_v} - 2\mathcal{C}_{K_v}$ is an elliptic operator on $Z_v \backslash G_v$. Note that here we calculate \mathcal{C}_{K_v} by using the Killing form of $\text{Lie}(Z_v \backslash G_v)$ instead of its own Killing form.

2.2. L -function Theory for K -finite Vectors. The proof of the fact that the representation of $G(\mathbb{A})$ on $L_0^2(G(F)\backslash G(\mathbb{A}), \omega)$ decomposes as a discrete direct sum of irreducible representations, as in Lemma 5.2 of [17], actually gives important information on K -finite vectors in an irreducible component π . They consequently have representatives in the space of smooth functions on the automorphic quotient, and are rapidly decreasing in any Siegel domain (Lemma 5.6 of [17]). Let the superscript “fin” mean “ K -finite”. The rapid decay is important, because it adds to the description of W_π^{fin} , the image of $\pi^{\text{fin}} \subset \pi \subset L_0^2(G(F)\backslash G(\mathbb{A}), \omega)$ under the Whittaker intertwiner

$$(2.1) \quad \varphi \mapsto W_\varphi(g) = \int_{F\backslash\mathbb{A}} \varphi(n(x)g)\psi(-x)dx$$

the important growth property, which is essential for the uniqueness of Whittaker model at archimedean places (Section 2.8 and 4.4 of [2] for local uniqueness, Section 3.5 of [2] for global uniqueness). If φ has a prescribed K -type and is a pure tensor, i.e. $W_\varphi(g) = \prod_v W_{\varphi,v}(g_v)$ splits, $W_{\varphi,v}(a(y)k)$ is forced to have rapid decay at ∞ , thus has nice behavior around 0

$$(2.2) \quad |W_{\varphi,v}(a(y)k)| \ll |y|_v^{1/2-\theta}.$$

Now let χ be a character of $F^\times \backslash \mathbb{A}^\times$ and $s \in \mathbb{C}$. Jacquet-Langlands [22] defined a functional on π^{fin} , called the (global) zeta-functional :

$$\zeta(s, \varphi, \chi) = \int_{F^\times \backslash \mathbb{A}^\times} \varphi(a(y))\chi(y)|y|^{s-1/2}d^\times y, \forall \varphi \in \pi, a(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\varphi(a(y))$ is rapidly decreasing at ∞ , it is also rapidly decreasing at 0 since

$$\varphi(a(y)) = \varphi(wa(y)) = \omega(y) \cdot w \cdot \varphi(a(y^{-1})), w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

Thus $\zeta(s, \varphi, \chi)$ is well defined for all s , and the functional equation characterizes the left invariance by w of φ

$$(2.3) \quad \zeta(s, \varphi, \chi) = \zeta(1-s, w \cdot \varphi, \omega^{-1}\chi^{-1}).$$

If φ is a pure tensor in $\pi^{\text{fin}} \simeq \otimes'_v \pi_v^{\text{fin}}$, i.e. W_φ factorizes, then since

$$(2.4) \quad \varphi(g) = \sum_{t \in F^\times} W_\varphi(a(t)g),$$

we can see

$$\zeta(s, \varphi, \chi) = \prod_v \zeta(s, W_{\varphi,v}, \chi_v, \psi_v), \Re(s) > 1 + \theta$$

with

$$\zeta(s, W_{\varphi,v}, \chi_v, \psi_v) = \int_{F_v^\times} W_{\varphi,v}(a(y))\chi(y)|y|^{s-1/2}d^\times y.$$

The convergence is justified by the above local growth property of $W_{\varphi,v}$ and the fact that on an unramified finite place v , the local zeta-function equals

$$\zeta(s, W_{\varphi,v}, \chi_v, \psi_v) = L(s, \pi_v \otimes \chi_v) = (1 - \mu_v \chi_v(\varpi_v)q_v^{-s})^{-1}(1 - \nu_v \chi_v(\varpi_v)q_v^{-s})^{-1}$$

where $\pi_v = \text{Ind}_{B(F_v)}^{G(F_v)}(\mu_v, \nu_v)$ determines μ_v, ν_v .

The analysis of local zeta-functions shows that $\zeta(s, W_{\varphi,v}, \chi_v, \psi_v)$, as $W_{\varphi,v}$ varies over $W_{\pi,v}^{\text{fin}}$, have a “common divisor” $L(s, \pi_v \otimes \chi_v)$, which is a meromorphic function

on s such that $\frac{\zeta(s, W_{\varphi, v}, \chi_v, \psi_v)}{L(s, \pi_v \otimes \chi_v)}$, originally defined for $\Re(s) > \theta$, can be analytically continued into an entire function on $s \in \mathbb{C}$, and equals 1 for almost all places v . Furthermore, there is a functional equation

$$(2.5) \quad \frac{\zeta(s, W_{\varphi, v}, \chi_v, \psi_v)}{L(s, \pi_v \otimes \chi_v)} \epsilon(s, \pi_v, \chi_v, \psi_v) = \frac{\zeta(1-s, wW_{\varphi, v}, \omega_v^{-1}\chi_v^{-1}, \psi_v)}{L(1-s, \pi_v \otimes \omega_v^{-1}\chi_v^{-1})}$$

where $\epsilon(s, \pi_v, \chi_v, \psi_v)$ is an entire function of exponential type. Define usual and completed L -functions as

$$L(s, \pi \otimes \chi) = \prod_{v < \infty} L(s, \pi_v \otimes \chi_v), \Lambda(s, \pi \otimes \chi) = \prod_v L(s, \pi_v \otimes \chi_v), \Re(s) > 1 + \theta$$

then the analytic continuations and functional equations of these L -functions follow from the well-definedness of $\zeta(s, \varphi, \chi)$ and (2.3), (2.5). The identity

$$\zeta(s, \varphi, \chi) = L(s, \pi \otimes \chi) \prod_{v|\infty} \zeta(s, W_{\varphi, v}, \chi_v, \psi_v) \prod_{v < \infty} \frac{\zeta(s, W_{\varphi, v}, \chi_v, \psi_v)}{L(s, \pi_v \otimes \chi_v)}$$

can be evaluated at $s = 1/2$ without analytic continuation of any integral. Thus

$$(2.6) \quad L(1/2, \pi \otimes \chi) = \prod_{v|\infty} \zeta(1/2, W_{\varphi, v}, \chi_v, \psi_v)^{-1} \cdot \prod_{v < \infty} \frac{L(1/2, \pi_v \otimes \chi_v)}{\zeta(1/2, W_{\varphi, v}, \chi_v, \psi_v)} \cdot \zeta(1/2, \varphi, \chi).$$

Remark 2.1. *In fact, the above theory is valid for smooth (not necessarily K -finite) vectors as we explain in the next sections.*

2.3. Smooth Vectors in Different Models. For any Lie group G and a unitary representation (ρ, V) of G , let ρ^∞ be the subspace of smooth vectors in V . This is naturally a Fréchet space, defined by the semi-norms $\|X.v\|, X \in U(\mathfrak{g})$. If $V \subset L^2(M)$ is realized as a space of functions on a orientable real manifold M equipped with a smooth (right) G -action, and with a G -invariant volume form, then we can talk about Sobolev functions for the action. Note that the action $\rho : G \rightarrow U(V)$ need not coincide with the regular representation on $L^2(M)$ induced by the action of G on M . One may think about $\rho = \pi(\mu_1, \mu_2)$ in the principal unitary series of $G = GL_2(\mathbb{R})$.

Definition 2.2. *With the above notations, a function f on M is called Sobolev (for the G -action), if it is smooth for the differential structure of M , and if its class $[f]$ in $V \subset L^2(M)$ is a smooth vector. We write V^∞ or $\rho^{\text{nam}, \infty}$, if nam is the name of the model, or just ρ^∞ if the underlying model is clear, for the space of Sobolev functions.*

We obviously have $[\rho^{\text{nam}, \infty}] \subset \rho^\infty$. Reciprocally

Lemma 2.3. *Assume that:*

1. *For any $p \in M$, the map $s_p : G \rightarrow M, g \mapsto p.g$ is a submersion at the identity $e \in G$.*
2. *The action of any element $X \in \mathfrak{g}$ on $V \cap C^\infty(M)$ corresponds to a smooth vector field $v(X)$ on M .*

Then every vector $v \in \rho^\infty \subset L^2(M)$ has a representative in $C^\infty(M)$.

Definition 2.4. Fix a basis \mathcal{B} of \mathfrak{g} , for any positive integer $d > 0$, one can define a Sobolev norm on ρ^∞ by

$$S_d^\rho(v) = \max_{X_i \in \mathcal{B}, l \leq d} \|X_1 \dots X_l.v\|.$$

In fact, since the condition and the conclusion are of local nature, one may interpret everything on the open set C_p of some euclidean space, diffeomorphic to some open neighborhood U_p of some point $p \in M$. The assumptions 1,2 ensures that the Sobolev norms S_d^ρ are equivalent to the usual Sobolev norms on C_p in the underlying euclidean space. One can apply the classical Sobolev embedding theorem.

Corollary 2.5. Under the assumptions of the above lemma, for any $p \in M$, there is an integer d s.t. $\forall f \in L^2(M) \cap \rho^\infty$,

$$\sup_{q \in U_p} |f(q)| \ll_{p, U_p} S_d^\rho([f]).$$

The assumptions of the above lemma apply in the following situations:

- $\rho \subset R$ is a subrepresentation of the right regular representation on automorphic quotient space. In such a situation, we say that ρ is realized in the automorphic model: “aut”.
- $\rho = \pi$ is locally in principal unitary series with induced model, we say that ρ is realized in the induced model: “ind”.
- $\rho = W_\pi$ is the Whittaker model of a generic automorphic representation π . We say it is realized in the Whittaker model.
- $\rho = K_\pi$ is the Kirillov model of a generic automorphic representation π . We say it is realized in the Kirillov model.

Definition 2.6. If G is a totally disconnected group, acting on a totally disconnected space M , then a function f on M is said to be smooth, if it is locally constant on M and K -finite for any maximal compact subgroup K of G .

2.4. Smooth Vectors and Extended L -function Theory. We generalize the theory of L -function to smooth vectors. Use Corollary 2.5 and compactness of $F \backslash \mathbb{A}$, one may easily see (Corollary I.1.5 [12]) that the Whittaker functional

$$l : R^\infty \rightarrow \mathbb{C}, \varphi \mapsto W_\varphi(1)$$

is in the continuous dual space of R^∞ verifying

$$l(R(n(x)\varphi)) = \psi(x)l(\varphi)$$

and is related to the Whittaker intertwiner (2.1) by

$$W_\varphi(g) = l(R(g).\varphi).$$

When we restrict to an irreducible component π of R , or more precisely to $\otimes'_v \pi_v^\infty \subset \pi^\infty$, it splits as

$$l = \otimes'_v l_v$$

where l_v are local (continuous) Whittaker functionals of π_v^∞ verifying

$$l(n(x)w) = \psi_v(x)l(w), w \in \pi_v^\infty.$$

The study of $l_v, v < \infty$ is the same as in the K_v -finite case. So the uniqueness, the local functional equation (2.5), the rapid decay and the controlled behavior at 0 (2.2) remain valid. For a $v|\infty$, the uniqueness of l_v is established by Shalika [29]. So

one can define the smooth Whittaker model associated with a unitary irreducible representation π_v by

$$(2.7) \quad W_{\pi_v}^\infty = \{W_w(g) = l_v(\pi_v(g)w); w \in \pi_v^\infty\}$$

as well as its smooth Kirillov model

$$(2.8) \quad K_{\pi_v}^\infty = \{K_w(y) = W_w(a(y)); w \in \pi_v^\infty\}.$$

The rapid decay at infinity of the local Whittaker functions $W_w(g)$ can be found in Lemma I.1.2 [12]. Note that here, the rapid decay property is derived from the continuity of l_v . In fact, much more information is obtained by Jacquet, as a special case in Proposition 3.6 [11], where the behavior of $W_w(g)$ is completely characterized, which implies rapid decay and (2.2) in this situation. Consequently the rapid decay of $\varphi \in \otimes'_v \pi_v^\infty \subset \pi^\infty \subset R_0^\infty$ follows, by using (2.4). Furthermore, local functional equations (2.5) are obtained by Jacquet [21] with absolute convergence for $\Re(s) > \theta$ as in K_v -finite case.

Remark 2.7. *For a proof that rapid decay at infinity and local functional equation imply the controlled behavior at 0, see Proposition 3.2.3 [26].*

2.5. An Identification of Norms. A by-product of the above theory, already known in the K -finite case, is the identification of the norm on $\pi \subset R_0$ and the natural norm we put on global Whittaker models.

Lemma 2.8. *If $\pi = \hat{\otimes}'_v \pi_v \subset R_0$ and $\varphi \in \otimes'_v \pi_v^\infty$ is a pure tensor, then*

$$\|\varphi\|_{X(F)}^2 = \frac{(disc F)^{3/2} \Lambda^*(1, \pi \times \bar{\pi})}{\Lambda_F(2)} \prod_{v \in V_F} \frac{\zeta_v(2) \int_{F_v^\times \times K_v} |W_{\varphi, v}(a(y)k)|^2 d^\times y dk}{L(1, \pi_v \times \bar{\pi}_v)}$$

where Λ_F is the complete Dedekind zeta-function, $\Lambda(s, \pi \times \bar{\pi}) = \prod_{v \in V_F} L(s, \pi_v \times \bar{\pi}_v)$ is the completed L -function associated with $\pi \times \bar{\pi}$ and $\Lambda^*(1, \pi \times \bar{\pi})$ is its residue at 1.

Remark 2.9. *By [20], $C(\pi)^{-\epsilon} \ll L^*(1, \pi \times \bar{\pi}) \ll C(\pi)^\epsilon$. Here $C(\pi) = C_\infty(\pi)C_f(\pi)$ is the analytic conductor of π . $C_f(\pi)$ is the normal conductor. $L(s, \pi \times \bar{\pi}) = \prod_{v < \infty} L(s, \pi_v \times \bar{\pi}_v)$ is the incomplete Rankin-Selberg L -function.*

The proof of Lemma 2.8 is a standard use of Rankin-Selberg's method (c.f. [26] 4.4.2) : Unfold, for $\Re s \gg 1$

$$\int_{ZG(F) \backslash G(\mathbb{A})} \varphi(g) \bar{\varphi}(g) E(s, f)(g) d\bar{g}$$

to get

$$\begin{aligned} & \int_{\mathbb{A}^\times \times K} |W_\varphi(a(y)k)|^2 f_s(a(y)k) |y|^{-1} d^\times y dk \\ &= \int_{\mathbb{A}^\times \times K} |W_\varphi(a(y)k)|^2 |y|^{s-1} d^\times y dk \end{aligned}$$

where $f_s \in \pi(| \cdot |^{s-1/2}, | \cdot |^{1/2-s})$ is a spherical flat section, and

$$(2.9) \quad E(s, f)(g) = \sum_{\gamma \in B(F) \backslash G(F)} f_s(\gamma g).$$

Then take the residue at $s = 1$. In fact, $E(s, f)$ converges for $\Re(s) > 1$, has a meromorphic continuation to all $s \in \mathbb{C}$, and is of moderate growth for any given s (see for example section 3.7 of [2]). On an unramified place v , for a spherical $W_{\varphi, v}$, one has

$$(2.10) \quad \frac{\zeta_v(2s) \int_{F_v^\times \times K_v} |W_{\varphi, v}(a(y)k)|^2 |y|_v^{s-1} d^\times y dk}{L(s, \pi_v \times \bar{\pi}_v)} = |W_{\varphi, v}(1)|^2$$

which is 1 for a.e. v . The product $\prod_{v \in V_F} L(s, \pi_v \times \bar{\pi}_v)$ converges for $\Re(s) > 1$. Thus

$$\begin{aligned} \int_{ZG(F) \backslash G(\mathbb{A})} \varphi(g) \bar{\varphi}(g) E(s, f)(g) d\bar{g} &= \frac{(\text{disc} F)^{3/2}}{\Lambda_F(2s)} \Lambda(s, \pi, Ad) \\ &\cdot \prod_{v \in V_F} \frac{\zeta_v(2s) \int_{F_v^\times \times K_v} |W_{\varphi, v}(a(y)k)|^2 |y|_v^{s-1} d^\times y dk}{L(s, \pi_v \times \bar{\pi}_v)}, \Re s > 1. \end{aligned}$$

By the local behavior (2.2), one can evaluate the integrals on the right of $s = 1$. We can do more by taking into account the theory of Kirillov model.

$$\text{Define } B_1(F_v) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in F_v^\times, b \in F_v \right\}.$$

Proposition 2.10. *There are only two types of unitary irreducible representations of $B_1(F_v)$:*

1. *A character of $F_v^\times \simeq B_1(F_v)/N_v$.*
2. *The representation of $B_1(F_v)$ on $L^2(F_v^\times)$ defined by the formula : $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} f(x) = \psi(bx)f(ax)$, where ψ is a nontrivial character of F_v . Different ψ give equivalent representations. In particular, there is only one non one-dimensional unitary irreducible representation of $B_1(F_v)$. Let's denote this model by $\sigma(\psi)$.*

A rigorous proof of this proposition, in the case of an archimedean field, can be found in [25] Page 34 (29); and in the case of a non archimedean field, can be found in [8] Chapter 8.

We finally deduce:

Proposition 2.11. *Let π be the local component on v of a generic automorphic representation. For a $W \in W_\pi^\infty$ in the Whittaker model one actually has*

$$\int_{F_v^\times \times K_v} |W(a(y)k)|^2 d^\times y dk = \int_{F_v^\times} |W(a(y))|^2 d^\times y.$$

As a consequence, the formula in Lemma 2.8 becomes

$$\|\varphi\|_{X(F)}^2 = \frac{(\text{disc} F)^{1/2} \Lambda^*(1, \pi, Ad)}{\Lambda_F(2)} \prod_{v \in V_F} \frac{\zeta_v(2) \int_{F_v^\times} |W_{\varphi, v}(a(y))|^2 d^\times y}{L(1, \pi_v \times \bar{\pi}_v)}.$$

Remark 2.12. *The norm identifications actually justify the notations $W_{\pi_v}^\infty$ and $K_{\pi_v}^\infty$ as smooth vectors in their completions W_{π_v} and K_{π_v} .*

We have similar relation for Eisenstein series.

Lemma 2.13. *If $\pi = \pi(\chi_1, \chi_2)$ is (unitary) Eisenstein, and $\varphi(g) = E(0, f)(g)$ with $E(s, f)(g)$ defined as in (2.9), for some $f = \prod_v f_v \in \pi^{\text{ind}, \text{fin}}$ in the induced model, then one can define the Eisenstein norm of φ by*

$$\|\varphi\|_{Eis}^2 = \int_K |f(k)|^2 dk.$$

The following relation holds

$$(disc F)^{1/2} \prod_{v \in V_F} \frac{\zeta_v(2)}{\zeta_v(1)^2} \int_{F_v^\times} |W_{\varphi, v}(a(y))|^2 d^\times y = \|\varphi\|_{Eis}^2,$$

and the local data are defined as the analytic continuation in (χ_1, χ_2) of

$$W_{\varphi, v}(g) = W_{f, v}(g) = \int_{F_v} f_v(w_n(x)g) \psi_v(-x) dx.$$

Remark 2.14. *One can interpret $W_{\varphi, v}(a(y)) \chi_{2, v}(y)^{-1} |y|^{-1/2}$ as the Fourier transform of $x \mapsto f(w_n(x))$. The above norm identification is then a formal consequence of Plancherel formula as discussed in 3.1.6 of [26]. One can also use Theorem 4.6.5 of [2].*

2.6. Spectral Decomposition. The spectral decomposition, in the L^2 sense, is established in the first 4 sections of [19], which gives

$$(2.11) \quad R = \bigoplus_{\pi \text{ cuspidal}} \pi \oplus \int_{-i\infty}^{i\infty} \bigoplus_{\xi \in \widehat{F^\times \backslash \mathbb{A}^\times(1)}} \pi_{s, \xi} \frac{ds}{4\pi i} \oplus \bigoplus_{\chi \in \widehat{F^\times \backslash \mathbb{A}^\times}, \chi^2 = \omega} \chi \circ \det$$

where, $\pi_{s, \xi} = \pi(\xi \cdot |\cdot|^s, \omega \xi^{-1} \cdot |\cdot|^{-s})$. Note that $\pi_{s, \xi} \simeq \pi_{-s, \omega \xi^{-1}}$. According to Proposition I.1.4 of [11], the above spectral decomposition has an analogy for smooth vectors, namely

$$(2.12) \quad R^\infty = \bigoplus_{\pi \text{ cuspidal}} \pi^\infty \oplus \int_{-i\infty}^{i\infty} \bigoplus_{\xi \in \widehat{F^\times \backslash \mathbb{A}^\times(1)}} \pi_{s, \xi}^\infty \frac{ds}{4\pi i} \oplus \bigoplus_{\chi \in \widehat{F^\times \backslash \mathbb{A}^\times}, \chi^2 = \omega} \chi \circ \det$$

with convergence in the topology of R^∞ . We are going to establish

Theorem 2.15. *Suppose $\varphi \in R^\infty$, viewed as a function on $G(\mathbb{A})$, then the following decomposition*

$$\begin{aligned} \varphi(g) = & \sum_{\chi \in \widehat{F^\times \backslash \mathbb{A}^\times}, \chi^2 = \omega} \frac{\langle \varphi, \chi \circ \det \rangle}{\text{Vol}(X(F))} \chi \circ \det(g) + \sum_{\pi \text{ cuspidal}} \sum_{e \in \mathcal{B}(\pi)} \langle \varphi, e \rangle e(g) \\ & + \sum_{\xi \in \widehat{F^\times \backslash \mathbb{A}^\times(1)}} \sum_{\Phi \in \mathcal{B}(\pi_{s, \xi})} \int_{-i\infty}^{i\infty} \langle \varphi, E(s, \Phi) \rangle E(s, \Phi)(g) \frac{ds}{4\pi i} \end{aligned}$$

converges absolutely and uniformly on any compact subset, where $\mathcal{B}(?)$ means taking a basis of ? consisting of K -isotypical pure tensors. We may assume that if φ is $K_v[n_v]$ -invariant, then every function appearing at the right hand side is $K_v[n_v]$ -invariant for any finite place v . K_v need not be the standard maximal compact subgroup of G_v .

Remark 2.16. *Therefore, the sum $\sum_{\xi \in F^\times \setminus \widehat{\mathbb{A}}^{(1)}}$ is actually finite and the number depends only on F and n_v 's.*

If we consider the theory of Whittaker model as a theory of spectral decomposition with respect to the left action of $N(\mathbb{A})$, then we further have

Theorem 2.17. *Conditions are the same as in the above theorem. $\varphi \in R^\infty$, as functions on $G(\mathbb{A})$:*

$$\begin{aligned} \varphi(g) = & \varphi_N(g) + \sum_{\pi \text{ cuspidal}} \sum_{e \in \mathcal{B}(\pi)} \langle \varphi, e \rangle \sum_{\alpha \in F^\times} W_e(a(\alpha)g) + \\ & \sum_{\xi \in F^\times \setminus \widehat{\mathbb{A}}^{(1)}} \sum_{\Phi \in \mathcal{B}(\pi_{s,\xi})} \int_{-i\infty}^{i\infty} \langle \varphi, E(s, \Phi) \rangle \sum_{\alpha \in F^\times} W_{\Phi,s}(a(\alpha)g) \frac{ds}{4\pi i} \end{aligned}$$

converges absolutely and uniformly on any given Siegel domain.

Remark 2.18. *In practice, the basis $\mathcal{B}(?)$ will be chosen so that the components of its elements at some archimedean place v are K_v -isotypic where K_v is the standard maximal compact subgroup of G_v .*

We begin with some local Sobolev type analysis.

2.6.1. Local Bounds of K -isotypical Functions.

Lemma 2.19. *Let v be a finite place, and π be a unitary irreducible representation of G_v . Suppose $W \in W_\pi^\infty$, the smooth Whittaker model of π w.r.t. ψ_v , is invariant by $K_v[m]$, then we have the following Sobolev inequality*

$$|W(na(y)k)|^2 \text{Vol}(1 + \varpi_v^m \mathcal{O}_v) \leq \|W\|^2 1_{v(y) \geq v(\psi) - m}, n \in N_v, y \in F_v^\times, k \in K_v$$

with the convention $1 + \varpi_v^0 \mathcal{O}_v = \mathcal{O}_v^\times$. On an unramified place ($m = 0$), recall that

$$W(na(\varpi_v^l)k) = q_v^{-l/2} \frac{\alpha_1^{l+1} - \alpha_2^{l+1}}{\alpha_1 - \alpha_2} 1_{l \geq 0}$$

for some $|\alpha_1 \alpha_2| = 1, q_v^{-\theta} \leq |\alpha_1| \leq q_v^\theta$

We leave the proof to the reader.

Lemma 2.20. *Let v be a real place, and π be a unitary irreducible representation of G_v with central character ω . If $W \in W_\pi^\infty$, then*

$$\forall n \in N(\mathbb{R}), y \in \mathbb{R}^\times, k \in SO_2(\mathbb{R}), N \equiv 1 \pmod{2}, N > 0$$

$$|W(na(y)k)| \ll_{N,\epsilon} |y|^{-N} \max(|y|^\epsilon, |y|^{-1}) S_{N+1}^\pi(W).$$

Suppose further $W \in W_\pi^{\text{fin}}$ transforms under the action of $K_v = SO_2(\mathbb{R})$ according to the character

$$\kappa_\alpha = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \mapsto e^{im\alpha}.$$

Then we have the following Sobolev inequality, uniform in m ,

$$|W(na(y)k)| \ll_{N,\omega,\theta} |y|^{-N} \max(|y|^\epsilon, |y|^{-1}) \lambda_W^{N'} \|W\|$$

where λ_W is the eigenvalue for W of the elliptic operator $\Delta_v = -\mathcal{C}_{G_v} + 2\mathcal{C}_{K_v}$, and N' depends only on N and ω .

Let $U = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ be elements in the Lie algebra of $GL_2(\mathbb{R})$, then

$$T.W(a(y)) = -2\pi iyW(a(y)), U.W(a(y)) = y \frac{\partial}{\partial y} W(a(y)).$$

We may only consider the case $y \in \mathbb{R}_+^\times$. Then for $\forall x, y \in \mathbb{R}_+^\times$, we have

$$(-2\pi iy)^N W(a(y)) = T^N.W(a(x)) + \int_x^y UT^N.W(a(u))d^\times u.$$

Note that

$$\begin{aligned} \left| \int_x^y UT^N.W(a(u))d^\times u \right| &\leq \left(\int_x^y |UT^N.W(a(u))|^2 d^\times u \right)^{1/2} \left(\int_x^y d^\times u \right)^{1/2} \\ &\leq \|UT^N.W\| |\log(y/x)|^{1/2}. \end{aligned}$$

Thus

$$|(-2\pi iy)^N W(a(y))| \leq |T^N.W(a(x))| + \|UT^N.W\| |\log(y/x)|^{1/2}.$$

Integrating against $\min(x, 1/x)dx/x$ for $0 < x < \infty$, using Cauchy-Schwarz and $\sqrt{1/2}(\sqrt{a} + \sqrt{b}) \leq \sqrt{a+b}$, we get

$$2|(-2\pi iy)^N W(a(y))| \leq \|T^N.W\| + \|UT^N.W\| \int_0^\infty \min(x, 1/x) |\log(y/x)|^{1/2} d^\times x.$$

Using the bound $|\log t| \ll_\epsilon \max(t^\epsilon, t^{-\epsilon})$, we get

$$2|(-2\pi iy)^N W(a(y))| \ll_\epsilon \|T^N.W\| + \|UT^N.W\| \max(|y|^\epsilon, |y|^{-1}).$$

Thus the first inequality follows for $k = 1$. The general case follows by noting $S_{N+1}^\pi(k.W) \ll_N S_{N+1}^\pi(W)$, since the adjoint action of K on \mathfrak{g} has bounded coefficients.

The second follows from the equivalence of two system of Sobolev norms, one is S_d^π 's, the other is defined with Δ_v and $I \in Z(\mathfrak{g})$. The proof is technical. We give it in the next section (Theorem 2.29).

Before proceeding to the complex place case, let's first recall that the irreducible representations of $SU_2(\mathbb{C})$ are parametrized by $m \in \mathbb{N}$, denoted by (ρ_m, V_m) . Here V_m is the space of homogeneous polynomials in $\mathbb{C}[z_1, z_2]$ of degree $m+1$, equipped with the inner product

$$\langle P_1, P_2 \rangle = \int_{|z_1|^2 + |z_2|^2 \leq 1} P_1(z_1, z_2) \overline{P_2(z_1, z_2)} dz_1 dz_2.$$

The action of $SU_2(\mathbb{C})$ is given by

$$u.P(z_1, z_2) = P((z_1, z_2).u).$$

Let $P_{m,k}(z_1, z_2)$ be a multiple of $z_1^{m-k} z_2^k$, normalized s.t. they form an orthonormal basis of V_m . Now let π be a unitary irreducible representation of $G(\mathbb{C})$. Let $W_{m,k} \in W_\pi^{\text{fin}}$ span the ρ_m -isotypical subspace, with $W_{m,k}$ corresponding to $P_{m,k}$. Since ρ_m is unitary, we have the following relation

$$\sum_{k=0}^m |W_{m,k}(gu)|^2 = \sum_{k=0}^m |W_{m,k}(g)|^2, \forall u \in SU_2(\mathbb{C}).$$

Therefore, we only need to bound $W_{m,k}(a(y))$ in order to bound $W_{m,k}(g)$. This works exactly as in the real place case. We omit the proof.

Lemma 2.21. *Let v be a complex place, and π be a unitary irreducible representation of G_v with central character ω . If $W \in W_\pi^\infty$, then*

$$\forall n \in N(\mathbb{C}), y \in \mathbb{C}^\times, k \in SU_2(\mathbb{C}), N \in \mathbb{N}$$

$$|W(na(y)k)| \ll_{N,\epsilon} |y|_v^{-N} \max(|y|_v^\epsilon, |y|_v^{-1/2}) S_{2N+2}^\pi(W).$$

Suppose further $W \in W_\pi^{\text{fin}}$ transforms under the action of $K_v = SU_2(\mathbb{C})$ according to ρ_m and corresponds to some $P_{m,k}$. Then we have the following Sobolev inequality, uniformly in m ,

$$|W(na(y)k)| \ll_{N,\omega,\theta} |y|_v^{-N} \max(|y|_v^\epsilon, |y|_v^{-1/2}) \lambda_W^{N'} \|W\|$$

where λ_W is the eigenvalue for W of the elliptic operator $\Delta_v = -\mathcal{C}_{G_v} + 2\mathcal{C}_{K_v}$, and N' depends only on N and ω .

2.6.2. Proof of Theorems 2.15, 2.17. We first deal with the cuspidal parts in the equations of Theorems 2.15, 2.17.

Let $e \in \pi \subset R_0$ be a K -isotypic vector, with local Whittaker model $W_{e,v}$. Denote by n_v the K_v -type of $W_{e,v}$, i.e.

- if $v < \infty$, then $W_{e,v}$ is $K_v[n_v]$ -invariant. For a.e. v , $n_v = 0$.
- if v is a real place, then $W_{e,v}$ transforms under $SO_2(\mathbb{R})$ as $e^{in_v\alpha}$.
- if v is a complex place, then $W_{e,v}$ transforms under $SU_2(\mathbb{C})$ as some $P_{n_v,k}$.

Collecting all the estimations in the previous subsection, using Lemma 2.8 or Proposition 2.11 with $\|e\| = 1$ and $C_\infty(\pi) \ll \lambda_{e,\infty} = \prod_{v|\infty} \lambda_{e,v}$, $C_f(\pi) \leq \prod_{v<\infty} q_v^{n_v}$ we obtain

$$\begin{aligned} W_e(na(y)k) &\ll_{F,N,\epsilon} |y|_\infty^{-N} \lambda_{e,\infty}^{N'} \left(\prod_{v<\infty} q_v^{n_v} \right)^\epsilon \prod_{v<\infty, n_v \neq 0} L(1, \pi_v \times \bar{\pi}_v) \text{Vol}(1 + \varpi_v^{n_v} \mathcal{O}_v)^{-1} \\ &\quad \cdot \prod_{v<\infty} 1_{v(y) \geq v(\psi) - n_v}, \text{ where } |y|_\infty = \prod_{v|\infty} |y|_v. \end{aligned}$$

The term $\prod_{v<\infty, n_v \neq 0} L(1, \pi_v \times \bar{\pi}_v) \text{Vol}(1 + \varpi_v^{n_v} \mathcal{O}_v)^{-1}$ can be bounded from above by

a constant depending only on $n_v, v < \infty$, we thus get

$$W_e(na(y)k) \ll_{F,N,\epsilon,(n_v)_{v<\infty}} \lambda_{e,\infty}^{N'} |y|_\infty^{-N} \prod_{v<\infty} 1_{v(y) \geq v(\psi) - n_v}.$$

Now since

$$e(na(y)k) = \sum_{\alpha \in F^\times} W_e(a(\alpha)na(y)k) = \sum_{\alpha \in F^\times} W_e(n'a(\alpha y)k), n' = a(\alpha)na(\alpha)^{-1}$$

we have

$$\sum_{\alpha \in F^\times} |W_e(a(\alpha)na(y)k)| \ll_{F,N,\epsilon} C(n_v, v < \infty) \lambda_{e,\infty}^{N'} \sum_{\alpha \in F^\times} |\alpha y|_\infty^{-N} \prod_{v<\infty} 1_{v(\alpha y) \geq v(\psi) - n_v}.$$

Consider the splitting $\mathbb{A}^\times \simeq \mathbb{A}^1 \times \mathbb{R}_+$ and write $y = y_1 t$ s.t. $y_1 \in \mathbb{A}^1$ and $t \in \mathbb{R}_+ \hookrightarrow \mathbb{A}^\times$ with trivial finite components. We need only consider y_1 in a fundamental domain of $F^\times \backslash \mathbb{A}^1$. Since the quotient $F^\times \backslash \mathbb{A}^1$ is compact, we may assume that there exist $0 < c < C$ s.t. for any place v , $c \leq |y_{1,v}|_v \leq C$ and for a.e. v , say $\forall v > v_0$, $|y_{1,v}|_v = 1$. So the condition imposed in $\prod_{v<\infty}$ implies $|\alpha|_v \leq c^{-1} q_v^{n_v - v(\psi)}$

and $|\alpha|_v \leq 1, \forall v > v_0$ (one may choose v_0 big enough depending only on n_v 's) in

oder to get a non zero contribution. Thus, α runs over the non zero elements in a lattice of F_∞ depending only on n_v 's. Therefore

$$\sum_{\alpha \in F^\times} |\alpha y|_\infty^{-N} \prod_{v < \infty} 1_{v(\alpha y) \geq v(\psi) - n_v} \ll_{n_v, v < \infty} |y|_\infty^{-N} \ll_{F, N} |y|^{-N}.$$

We conclude

$$(2.13) \quad \sum_{\alpha \in F^\times} |W_e(a(\alpha)na(y)k)| \ll_{F, N, n_v, v < \infty} \lambda_{e, \infty}^{N'} |y|^{-N}.$$

Now let's turn to the Eisenstein parts of Theorems 2.15, 2.17.

Using Lemma 2.13 instead of 2.8 in the above argument, we get

$$(2.14) \quad \sum_{\alpha \in F^\times} |W_{\Phi, s}(a(\alpha)na(y)k)| \ll_{F, N, n_v, v < \infty} \lambda_{\Phi, s, \infty}^{N'} |y|^{-N}.$$

We have an expression for the constant term

$$E(s, \Phi)_N(g) = \Phi_s(g) + M(s)\Phi_s(g).$$

$\Phi_s|_K$ belongs to some irreducible component σ of $\text{Res}_K^{G(\mathbb{A})} \pi_{s, \xi} = \text{Ind}_{K \cap B(\mathbb{A})}^K(\xi, \omega \xi^{-1})$. From basic representation theory, it is easy to see that

$$\Phi_s(k) = \sqrt{\dim \sigma} < \sigma(k).v, v_0 >_\sigma, v, v_0 \in \sigma \text{ of norm } 1, \sigma(b).v_0 = (\xi, \omega \xi^{-1})(b).v_0.$$

Thus follows the bound ($\Re(s) = 0$)

$$|\Phi_s(na(y)k)| = |y|^{1/2} |\Phi_s(k)| \leq |y|^{1/2} \sqrt{\dim \sigma} \ll_{n_v, v < \infty} |y|^{1/2} \lambda_{K_\infty}(\Phi)^{1/2}$$

where $\lambda_{K_\infty}(\Phi)$ is the eigenvalue of Φ for the Casimir of K_∞ . Note that $M(s)$ is unitary for $s \in i\mathbb{R}$ and doesn't change the K -type, thus

$$|M(s)\Phi_s(na(y)k)| \ll_{n_v, v < \infty} |y|^{1/2} \lambda_{K_\infty}(\Phi)^{1/2}.$$

Hence

$$(2.15) \quad |E(s, \Phi)_N(na(y)k)| \ll_{n_v, v < \infty} |y|^{1/2} \lambda_{K_\infty}(\Phi)^{1/2} \leq |y|^{1/2} \lambda_{\Phi, s, \infty}^{1/2}.$$

Theorems 2.15 & 2.17 will be established using the following generalized Weyl's law

Theorem 2.22. *Given a sequence of non-negative integers $\bar{n} = (n_v)_{v < \infty}$ with $n_v = 0$ for a.e.v. Define*

$$K_{fin}[\bar{n}] = \prod_{v < \infty} K_v[n_v]$$

and consider the space $R^{K_{fin}[\bar{n}]} = L^2(G(F) \backslash G(\mathbb{A}), \omega)^{K_{fin}[\bar{n}]}$. It is actually a representation of $G(F_\infty) \times K_f$. The operator $\Delta_\infty = \prod_{v < \infty} \Delta_v$ is self-dual and commutes

with the action of K . We have $\Delta_\infty^{-1-\epsilon}$ is of trace class in $R^{K_{fin}[\bar{n}]}$. More precisely,

$$\begin{aligned} & \sum_{\pi'} \sum_e |\lambda_{e, \infty}|^{-1-\epsilon} + \sum_{\xi} \sum_{\Phi} \int_{-\infty}^{\infty} |\lambda_{\Phi_{i\tau}, \infty}|^{-1-\epsilon} \frac{d\tau}{2\pi} \\ &= O_\epsilon(\text{Vol}(Z(\mathbb{A})G(F) \backslash G(\mathbb{A})/K_{fin}[\bar{n}])). \end{aligned}$$

Here $\lambda_{e, \infty}$ runs over the discrete spectrum of Δ_∞ , and $\lambda_{\Phi_{i\tau}, \infty}$ runs over the continuous spectrum of Δ_∞ .

Remark 2.23. *We only need a weaker version here. Namely, we only need Δ_∞^{-N} to be of trace class for some $N > 0$.*

Remark 2.24. *If instead of $K_{fin}[\bar{n}]$ we consider $K_\infty \times K_{fin}[\bar{n}]$, the above theorem would coincide with the traditional geometrical Weyl's law. Note that this kind of Weyl's law was already used to establish theorems like 2.15 for K_∞ -fixed case, e.g. [15]. Weyl's law is at the heart of the theory of analytical spectral decomposition.*

Remark 2.25. *This theorem will appear as a part of the Ph.D thesis of Marc R. Palm at Göttingen.*

Definition 2.26. (c.f. page 292 [9]) *The Schwartz function space R^s is the space of smooth functions φ in $\text{Ind}_{Z(\mathbb{A})G(F)}^{G(\mathbb{A})}\omega$, which are rapidly decreasing in any given Siegel domain, as well as $X\varphi$ for any $X \in U(\mathfrak{g})$.*

The above argument also gives

Corollary 2.27. *We have $R_0^\infty \subset R^s \subset R^\infty$.*

Remark 2.28. *If we take into account the central character, namely, if we write R_ω instead of R , we have $R_\omega^s R_{\omega'}^s \subset R_{\omega\omega'}^s$. In particular, if the central character is the trivial character ω_0 , $R_{\omega_0}^s$ is a ring for the pointwise multiplication.*

2.7. Two Sobolev Norm Systems. Let v be an archimedean place, and π be a unitary irreducible representation of G_v with a fixed central character ω . Let $\{I_1, \dots, I_r\}$ be a basis of $Z(\mathfrak{g}_v)$. In our case, $r = 1$ if v is a real place, and $r = 2$ if v is a complex place. We define the Sobolev norm system

$$H_d^\pi(v) = \max_{i_1 + \dots + i_r + 2j = d} \|I_1^{i_1} \cdots I_r^{i_r} \Delta_v^j v\|.$$

Theorem 2.29. *The Sobolev norm system H_d^π is equivalent to the Sobolev system S_d^π for π a local component of an automorphic representation. If the parameter s of π belongs to $i\mathbb{R} \cup [-\theta, \theta]$ with $\theta < 1/2$, then the implicit constants in the above equivalence depend only on θ .*

The rest of this section is devoted to the proof of Theorem 2.29.

2.7.1. v is a real place. The Hecke algebra $\mathcal{H}_v = U(\mathfrak{g}) \oplus \underline{\epsilon} * U(\mathfrak{g})$, where $\underline{\epsilon}$ is the Dirac measure at $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. There is a classification of unitarizable irreducible (\mathcal{H}_v, K_v) modules (c.f. for example 4.A [17]). Each of them $\pi(\mu_1, \mu_2)$ is parametrized by $s_1, s_2 \in \mathbb{C}, m_1, m_2 \in \{0, 1\}$. Put $s = s_1 - s_2, t = s_1 + s_2 \in i\mathbb{R}, m = m_1 - m_2$. There are 3 different series: 1. $s \in i\mathbb{R}$; 2. $0 < s < 1$ but only $s < 2\theta$ is possible for the local component of an automorphic representation; 3. $0 < s = p \in \mathbb{Z}, s - m$ is an odd integer. In each case, there is an orthogonal, not necessarily normalized, basis consisting of K_v -isotypical vectors, $\{e_k\}$. In cases 1 and 2, k runs through $k \equiv m \pmod{2}$, and in the case 3, $|k| \geq p + 1, k \equiv p + 1 \pmod{2}$. There is a basis of $\mathfrak{g}_\mathbb{C}$, $\left\{ H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, V_+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, V_- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, J = id \right\}$ with explicit action given as

$$H.e_k = ike_k; V_+.e_k = (s + 1 + k)e_{k+2}; V_-.e_k = (s + 1 - k)e_{k-2}; J.e_k = te_k$$

$$\Delta_v.e_k = \left(\frac{1 - s^2}{8} + \frac{k^2}{4} \right) e_k.$$

Consider a general vector $v = \sum_k a_k e_k$, $a_k \in \mathbb{C}$. In the case 1, Theorem 2.6.2 of [2] implies $\|e_k\| = 1$. We easily deduce

$$\|H.v\|^2, \|V_+.v\|^2, \|V_-.v\|^2 \leq 16\|\Delta_v^{1/2}.v\|^2.$$

In the case 2, $\|e_k\|^2 = |\sqrt{\pi} \frac{\Gamma((s+1)/2)\Gamma(s/2)}{\Gamma((s+1+k)/2)\Gamma((s+1-k)/2)}|$ according to Theorem 2.6.4 of [2]. As a consequence

$$\frac{\|e_{k+2}\|^2}{\|e_k\|^2} = \left| \frac{s-1-k}{s+1+k} \right| \ll_{\theta} 1, \frac{\|e_{k-2}\|^2}{\|e_k\|^2} = \left| \frac{s-1+k}{s+1-k} \right| \ll_{\theta} 1.$$

We get

$$\|H.v\|^2, \|V_+.v\|^2, \|V_-.v\|^2 \ll_{\theta} 16\|\Delta_v^{1/2}.v\|^2.$$

In the case 3, it can be inferred from Theorem 2.6.5 of [2] that $\pi(\mu_1, \mu_2)$ has the following model: Let \mathbb{H}^+ be the Poincaré half plane, and \mathbb{H}^- be its opposite. The space is, with the coordinates $z = x + iy$

$$L^2(\mathbb{H}^{\pm}) = \left\{ f : \mathbb{H}^{\pm} \rightarrow \mathbb{C}, \text{ holomorphic} : \int_{y \neq 0} |f(z)|^2 y^{p+1} \frac{dx dy}{|y|^2} < \infty \right\}.$$

Therefore one may take, for $|k| \geq p+1$

$$e_k(z) = (z-i)^{-(k+p+1)/2} (z+i)^{(k-p-1)/2} 1_{\text{sgn}(k)\text{sgn}(y) < 0}.$$

Change to the Poincaré disk model, one calculates easily, with $B(\cdot, \cdot)$ the Beta function

$$\|e_k\|^2 = \pi 4^{-p} B((|k| - p - 1)/2 + 1, p).$$

Consequently

$$\frac{\|e_{k+2}\|^2}{\|e_k\|^2} \ll k^2, \frac{\|e_{k-2}\|^2}{\|e_k\|^2} \ll k^2$$

$$\|H.v\|^2, \|V_+.v\|^2, \|V_-.v\|^2 \ll 16\|\Delta_v.v\|^2.$$

We conclude that in all cases

$$\|H.v\|^2, \|V_+.v\|^2, \|V_-.v\|^2 \ll_{\theta} \|\Delta_v.v\|^2$$

thus $S_d^{\pi} \ll_{\theta, d} H_d^{\pi} \ll S_{2d}^{\pi}$, and the two systems are equivalent.

2.7.2. v is a complex place. The unitary irreducible series $\pi(\mu_1, \mu_2)$ are parametrized by $s_1, s_2 \in \mathbb{C}, k_1, k_2 \in \mathbb{Z}$ with $t = s_1 + s_2 \in i\mathbb{R}, s = s_1 - s_2 \in i\mathbb{R}$ and $\mu_j(\rho e^{i\alpha}) = \rho^{2s_j} e^{ik_j\theta}, j = 1, 2$. Or $t = s_1 + s_2 \in i\mathbb{R}, 0 < s = s_1 - s_2 < 2\theta, k_1 = k_2$. Let $n_0 = k_1 - k_2$. We may suppose $n_0 \geq 0$ after exchange μ_1 and μ_2 if necessary. The representation $\pi(\mu_1, \mu_2)$ has an orthogonal basis $\left\{ e_{n,k}^{(n_0)} : 0 \leq k \leq n, n \geq |n_0|, n \equiv |n_0| \pmod{2} \right\}$ determined by

$$e_{n,k}^{(n_0)} \left(\begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} g \right) = \mu_1(y_1) \mu_2(y_2) |y_1/y_2| e_{n,k}^{(n_0)}(g), \forall g \in G_v$$

$$e_{n,k}^{(n_0)} \left(\begin{pmatrix} e^{i\alpha_1} & 0 \\ 0 & e^{-i\alpha_1} \end{pmatrix} u \begin{pmatrix} e^{i\alpha_2} & 0 \\ 0 & e^{-i\alpha_2} \end{pmatrix} \right) = e^{in_0\alpha_1} e^{i(n-2k)\alpha_2}, \forall u \in K_v = SU_2(\mathbb{C})$$

$$e_{n,k}^{(n_0)} \left(\begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \right) = (\cos \beta)^{\frac{n+n_0}{2}-k} (\sin \beta)^{k-\frac{n-n_0}{2}} P^{\left(\frac{n_0-n}{2}+k, \frac{n_0+n}{2}-k\right)}_{\frac{n-n_0}{2}} (\cos 2\beta)$$

where $P_k^{(\alpha, \beta)}$ are the Jacobi polynomials. Alternatively,

$$e_{n,k}^{(n_0)} = \frac{\langle \rho_n(u) z_1^{n-k} z_2^k, z_1^{n-k_0} z_2^{k_0} \rangle_{\rho_n}}{\langle z_1^{n-k_0} z_2^{k_0}, z_1^{n-k_0} z_2^{k_0} \rangle_{\rho_n}}, n - 2k_0 = n_0.$$

It will also be convenient to extend by 0 to all integers n, k . The (complexified) Lie algebra su_2 has a basis

$$H_2 = \begin{pmatrix} i & \\ & -i \end{pmatrix}, X_{\pm} = \pm \begin{pmatrix} & -1/2 \\ 1/2 & \end{pmatrix} - i \begin{pmatrix} & i/2 \\ i/2 & \end{pmatrix}$$

which act as

$$\begin{aligned} H_2 \cdot e_{n,k}^{(n_0)} &= i(n-2k)e_{n,k}^{(n_0)}, X_+ \cdot e_{n,k}^{(n_0)} = (n-k)e_{n,k+1}^{(n_0)}, X_- \cdot e_{n,k}^{(n_0)} = ke_{n,k-1}^{(n_0)} \\ \Delta_v \cdot e_{n,k}^{(n_0)} &= ((1-s^2-n_0^2)/8 + n(n+2)/4)e_{n,k}^{(n_0)}. \end{aligned}$$

It is obvious then that $\Delta_v^{-1-\epsilon}$ is of trace class in $\pi(\mu_1, \mu_2)$. A standard argument then shows that it suffices to prove Theorem 2.29 for v running over an orthonormal basis. The Cartan complement \mathfrak{p} of su_2 has a basis (We ignore the center)

$$H_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, Y_+ = ad(X_+)(H_1), Y_- = ad(X_-)(H_1).$$

Using the recurrence relations of Jacobi polynomials (c.f. [1]), we can find

$$\begin{aligned} H_1 \cdot e_{n,k}^{(n_0)} &= \frac{2(k+1)(k+n_0+1)(2k+n_0+2+2s)}{(2k+n_0+1)(2k+n_0+2)} e_{n+2,k+1}^{(n_0)} \\ &\quad + \frac{2sn_0(n-2k)}{(2k+n_0)(2k+n_0+2)} e_{n,k}^{(n_0)} + \frac{(n+n_0)(4k-n+n_0)(2s-2k+n_0)}{2(2k+n_0)(2k+n_0+1)} e_{n-2,k-1}^{(n_0)}. \end{aligned}$$

Since

$$Y_+ \cdot e_{n,k}^{(n_0)} = X_+ H_1 \cdot e_{n,k}^{(n_0)} - H_1 X_+ \cdot e_{n,k}^{(n_0)}, Y_- \cdot e_{n,k}^{(n_0)} = X_- H_1 \cdot e_{n,k}^{(n_0)} - H_1 X_- \cdot e_{n,k}^{(n_0)}$$

we can only consider the actions of H_1, H_2, X_+, X_- if we don't want to optimize.

Case 1: $s \in i\mathbb{R}$. Then we are in the unitary principal series case and the norm structure is the standard L^2 norm on $SU_2(\mathbb{C})$.

$$\|e_{n,k}^{(n_0)}\|^2 = \frac{(n-k)!k!}{(\frac{n-n_0}{2})! (\frac{n+n_0}{2})! (n+1)}.$$

One easily verifies $\|X \cdot e_{n,k}^{(n_0)}\| \ll \|\Delta_v^{3/2} \cdot e_{n,k}^{(n_0)}\|$, $X = H_1, H_2, X_+, X_-$, hence

$$\|X \cdot v\| \ll \|\Delta_v^4 \cdot v\|, \forall v \in \pi^\infty, X = H_1, H_2, X_{\pm}, Y_{\pm}.$$

Case 2: $0 < s < 2\theta < 1$. Then $n_0 = 0$, thus $n \equiv 0 \pmod{2}$. Let's write $e_{n,k}^{(s,0)} = e_{n,k}^{(0)}$ to emphasize the dependence on s . The norm structure is defined via the intertwining operator (with analytic continuation for $s < 0$)

$$M(s)e_{n,k}^{(s,0)}(g) = \int_{\mathbb{C}} e_{n,k}^{(s,0)}(n(x)g)dx = \lambda_{n,k}(s)e_{n,k}^{(-s,0)}(g).$$

Lemma 2.30. $\lambda_{n,k}(s) = (-1)^{n/2} \pi \frac{(s-1) \cdots (s-n/2)}{s(s+1) \cdots (s+n/2)}$ Therefore,

$$\|e_{n,k}^{(s,0)}\|^2 = (-1)^{n/2} \pi \frac{(s-1) \cdots (s-n/2)}{s(s+1) \cdots (s+n/2)} \frac{(n-k)!k!}{(\frac{n}{2})! (\frac{n}{2})! (n+1)}$$

With this, we easily see

$$\|X.v\| \ll_{\theta} \|\Delta_v^4.v\|, \forall v \in \pi^{\infty}, X = H_1, H_2, X_{\pm}, Y_{\pm}.$$

To prove the lemma, first consider $n = 2k$. We know $e_{2k,k}^{(s,0)}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = P_k^{(0,0)}(1) = 1$, so

$$\lambda_{2k,k}(s) = M(s)e_{n,k}^{(s,0)}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \frac{\pi}{2} \int_{-1}^1 \left(\frac{1-t}{2}\right)^{s-1} P_k^{(0,0)}(t) dt.$$

Now we can use the recurrence relation of Legendre polynomials to establish

$$\lambda_{2k+2,k+1}(s) = \frac{2(2k+1)}{s} \lambda_{2k,k}(s+1) + \lambda_{2k-2,k-1}(s).$$

The first two values are easy to obtain $\lambda_{0,0}(s) = \pi/s$, $\lambda_{2,1}(s) = -\frac{\pi(s-1)}{s(s+1)}$. By induction, we get

$$\lambda_{2k,k}(s) = (-1)^k \pi \frac{(s-1) \cdots (s-k)}{s(s+1) \cdots (s+k)}.$$

Since $M(s)$ commutes with the action of G_v , it commutes with the action of X_+, X_- . It follows that $\lambda_{n,k}(s) = \lambda_{n,n/2}(s), \forall k$. This proves the above lemma and concludes the proof of Theorem 2.29.

2.8. Construction of Automorphic Forms from Local Kirillov Models.

The norm identifications tell us that, given a pure tensor $\varphi \in \otimes'_v \pi_v^{\infty}$, resulting from (2.1), the $W_{\varphi,v}$ or the $K_{\varphi,v}$ must be a smooth vector in W_{π_v} or K_{π_v} . Conversely, if we are given $K_v \in K_{\pi_v}^{\infty}$, which uniquely determine corresponding $W_v \in W_{\pi_v}^{\infty}$, and form $W(g) = \prod_v W_v(g_v)$, and φ by (2.4), are we sure to get an element in

π^{∞} ? The converse theorem, as is discussed in the section 5.2 of [11], gives an affirmative answer. Note that, to determine W_v from K_v for an archimedean place v , a concrete way is to apply the Casimir element \mathcal{C} of $GL_2(\mathbb{R})$ in the real case, or the two embedded Casimir elements of $GL_2(\mathbb{R})$ in $GL_2(\mathbb{C})$ to get partial differential equations, since these elements should act as scalars depending only on π_v , then solve the corresponding Dirichlet problems.

Alternatively, maybe also more naturally and directly, if one wants to avoid the converse theorem, one may decompose W as an infinite sum of K -isotypical Whittaker functions, then change the order of summation to show that φ is a convergent (thanks to the local and global estimations in the above sections) infinite sum of K -isotypical functions in π , with rapidly decreasing spectral parameter for K , thus is itself in π^{∞} .

2.9. Decay of Matrix Coefficients. Consider a place v , let π_{λ} be the complementary series representation of G_v with parameter $\lambda/2$ and with trivial central character. It has a unique K_v invariant unit vector w^0 . The elementary spherical function associated with π_{λ} is defined to be (following Harish-Chandra's notation)

$$\varphi_{v,\lambda}(g) = \langle \pi_{\lambda}(g)w^0, w^0 \rangle.$$

Its limit when $\lambda \rightarrow 0$, denoted by $\varphi_{v,0} = \Xi_v$, is the Harish-Chandra function. They are all positive and bi- K_v -invariant.

Theorem 2.31. *Let π be any unitary irreducible representation of G_v . Let x_1, x_2 be 2 K_v -finite vectors in π . Then*

1 If π is tempered, then

$$\langle \pi(g)x_1, x_2 \rangle \leq \dim(K_v x_1)^{1/2} \dim(K_v x_2)^{1/2} \|x_1\| \cdot \|x_2\| \Xi_v(g).$$

2 If π is in the complementary series with parameter $\lambda/2$, then for any $\epsilon > 0$, there is a $A_v(\epsilon) > 0$

$$\langle \pi(g)x_1, x_2 \rangle \leq A_v(\epsilon) \dim(K_v x_1)^{1/2} \dim(K_v x_2)^{1/2} \|x_1\| \cdot \|x_2\| \Xi_v(g)^{1-\lambda-\epsilon}.$$

Here $\dim(K_v x) = \dim \text{span}(K_v \cdot x)$ is the dimension of the span of x by K_v action.

The tempered case is well known in [13]. The non-tempered case, first proved in Theorem 2.11 [30] for real case, then recaptured in Lemma 9.1 [31], essentially is based on the following estimation

$$(2.16) \quad A_v(\epsilon)^{-1} \varphi_{v,0}^{1-\lambda+\epsilon} \leq \varphi_{v,\lambda} \leq \varphi_{v,0}^{1-\lambda}.$$

This is an elementary exercise in analysis, we leave it to the reader.

3. OUTLINE OF THE PROOF: CUSPIDAL CASE

We first note, by the extended L -function theory that Theorem 1.1 can be divided into the following two parts:

Proposition 3.1. *There is a pure tensor $\varphi \in \otimes'_v \pi_v^\infty$ such that $\forall \epsilon > 0$*

$$(3.1) \quad \prod_{v|\infty} \zeta(1/2, W_{\varphi,v}, \chi_v, \psi_v)^{-1} \cdot \prod_{v<\infty} \frac{L(1/2, \pi_v \otimes \chi_v)}{\zeta(1/2, W_{\varphi,v}, \chi_v, \psi_v)} \ll_{\epsilon,F} Q^{1/2+\epsilon}$$

where $Q = C(\chi)$ is the analytic conductor of χ .

Proposition 3.2. *Suppose π is cuspidal. For the same φ , there is an absolute constant $\delta > 0$ such that $\forall \epsilon > 0$*

$$(3.2) \quad \zeta(1/2, \varphi, \chi) \ll_{\epsilon,\pi} Q^{-\delta+\epsilon}.$$

We may choose $\frac{1-2\theta}{8}$, or $\frac{25}{256}$ using the best known result of [4] i.e. $\theta = 7/64$.

The local estimation Proposition 3.1 will be a direct consequence of the choice of φ . The explicit choice will be given later. For the current discussion, we only need to know

1. $\varphi = n(T)\varphi_0$ with φ_0 a fixed pure tensor in π^∞ , $T \in \mathbb{A}$ and $|T| \in [Q^{1-\epsilon}, Q^{1+\epsilon}]$;
2. $\varphi_0 \in R^s$ as defined in Definition 2.26.

In order to simplify notations and for further convenience, another functional on automorphic representations should be introduced :

$$l^{|\cdot|^s}(\varphi) = \int_{F^\times \backslash \mathbb{A}^\times} \varphi(a(y)) \chi(y) |y|^s d^\times y.$$

Note that for π cuspidal one has

$$l^{|\cdot|^s}(\varphi) = \zeta(s+1/2, \varphi, \chi).$$

So (3.2) is equivalent to

$$l^X(\varphi) \ll_{\epsilon,\pi} Q^{-\delta+\epsilon}.$$

Let's restrict to the case where π is cuspidal. There is also a local analogue of this functional :

$$l^{\chi_v|\cdot|^s}(W_{\varphi,v}) = \int_{F_v^\times} W_{\varphi,v}(a(y))\chi(y)|y|^s d^\times y.$$

To deal with the fact that $F^\times \backslash \mathbb{A}^\times$ is non compact, we give below a truncation function $h \in C_c^\infty(\mathbb{R}_+)$ based on a fixed function h_0 such that h is supported in $[Q^{-\kappa-1}, Q^{\kappa-1}]$ and for $\forall \epsilon > 0$

$$l^\chi(\varphi) = \int_{F^\times \backslash \mathbb{A}^\times} h(|y|)\varphi(a(y))\chi(y)d^\times y + O_{h_0, \varphi_0, \epsilon}(Q^{-\kappa/2+\epsilon}).$$

Here, $\kappa \in (0, 1)$ is a parameter to be chosen optimally. Define another functional:

$$l^{\chi,h}(\varphi) = \int_{F^\times \backslash \mathbb{A}^\times} h(|y|)\varphi(a(y))\chi(y)d^\times y.$$

Consider the following average of Dirac measures :

$$\sigma = 1/M_E^2 \sum_{v, v' \in I_E} \delta_{a(|\varpi_v|_v |\varpi_{v'}^{-1}|_{v'})}$$

with

$$I_E = \{v \mid q_v \in [E, 2E], T_v = 0\}, M_E = |I_E| = O(E/\log E)$$

Here E is a suitable power of Q to be chosen optimally.

Lemma 3.3. *We have*

$$l^\chi(\varphi) = \int_{F^\times \backslash \mathbb{A}^\times} \sigma * h(|y|)\varphi(a(y))\chi(y)d^\times y + O_{h_0, \varphi_0, \epsilon}(Q^{-\kappa/2+\epsilon}).$$

The second step is to examine :

$$l^{\chi,h}(\sigma'_\chi * \varphi) = \int_{F^\times \backslash \mathbb{A}^\times} h(|y|)\sigma'_\chi * \varphi(a(y))\chi(y)d^\times y$$

where $\sigma'_\chi = 1/M_E^2 \sum_{v, v' \in I_E} \chi(\varpi_v \varpi_{v'}^{-1})\delta_{a(\varpi_v \varpi_{v'}^{-1})}$ is the adjoint measure of σ satisfying

$$l^{\chi,h}(\sigma'_\chi * \varphi) = \int_{F^\times \backslash \mathbb{A}^\times} \sigma * h(|y|)\varphi(a(y))\chi(y)d^\times y.$$

The inequality of Cauchy-Schwarz gives

$$(3.3) \quad |l^{\chi,h}(\sigma'_\chi * \varphi)|^2 \leq \int_{F^\times \backslash \mathbb{A}^\times} h(|y|)d^\times y \int_{F^\times \backslash \mathbb{A}^\times} |\sigma'_\chi * \varphi(a(y))|^2 h(|y|)d^\times y.$$

We then spectrally decompose $|\sigma'_\chi * \varphi_0|^2$ in $L^2(G(F) \backslash G(\mathbb{A}), 1)$ as in Theorem 2.17, which is possible because $\varphi_0 \in R^s$. Setting $l^h = l^{1,h}$ we have

$$(3.4) \quad l^h(n(T)|\sigma'_\chi * \varphi_0|^2) = l^h(n(T)|\sigma'_\chi * \varphi_0|_N^2)$$

$$(3.5) \quad + \sum_{\pi' \text{ cuspidal}} l^h(n(T)P_{\pi'}(|\sigma'_\chi * \varphi_0|^2))$$

$$(3.6) \quad + \frac{1}{4\pi} \sum_{\xi \in F^\times \backslash \mathbb{A}^{(1)}} \int_{-\infty}^{\infty} l^h(n(T)(P_{\xi, i\tau}(|\sigma'_\chi * \varphi_0|^2) - P_{\xi, i\tau}(|\sigma'_\chi * \varphi_0|_N^2)))d\tau$$

interchanging integrals being verified by Theorem 2.17. In every summand of (3.5) (resp. (3.6)) $P_{\pi'}$ (resp. $P_{\xi, i\tau}$) denotes the projection on the space of π' (resp. $\pi(\xi|\cdot|^{i\tau}, \xi^{-1}|\cdot|^{-i\tau})$). The function

$$|\sigma'_\chi * \varphi_0|^2 = \frac{1}{M_E^4} \sum_{v_1, v'_1, v_2, v'_2 \in I_E} \chi\left(\frac{\varpi_{v_1}}{\varpi_{v'_1}}\right) \chi^{-1}\left(\frac{\varpi_{v_2}}{\varpi_{v'_2}}\right) a\left(\frac{\varpi_{v_1}}{\varpi_{v'_1}}\right) \varphi_0 a\left(\frac{\varpi_{v_2}}{\varpi_{v'_2}}\right) \overline{\varphi}_0$$

Let's write

$$S_{cusp}(v_1, v'_1, v_2, v'_2) = \sum_{\pi' \text{ cuspidal}} l^h(n(T)P_{\pi'}(a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0)),$$

hence

$$(3.5) = \frac{1}{M_E^4} \sum_{v_1, v'_1, v_2, v'_2 \in I_E} \chi\left(\frac{\varpi_{v_1}}{\varpi_{v'_1}}\right) \chi^{-1}\left(\frac{\varpi_{v_2}}{\varpi_{v'_2}}\right) S_{cusp}(v_1, v'_1, v_2, v'_2).$$

Define

$$S_{cst}(v_1, v'_1, v_2, v'_2) = l^h(n(T)(a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0)_N) = l^h((a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0)_N),$$

$$\begin{aligned} S_{Eis}(v_1, v'_1, v_2, v'_2) &= \sum_{\xi \in F \times \backslash \mathbb{A}^{(1)}} \int_{-\infty}^{\infty} l^h(n(T)P_{\xi, i\tau}(a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0) \\ &\quad - P_{\xi, i\tau}(a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0)_N) d\tau. \end{aligned}$$

Therefore,

$$(3.7) \quad (3.4) = \frac{1}{M_E^4} \sum_{v_1, v'_1, v_2, v'_2 \in I_E} \chi\left(\frac{\varpi_{v_1}}{\varpi_{v'_1}}\right) \chi^{-1}\left(\frac{\varpi_{v_2}}{\varpi_{v'_2}}\right) S_{cst}(v_1, v'_1, v_2, v'_2),$$

$$(3.6) = \frac{1}{4\pi M_E^4} \sum_{v_1, v'_1, v_2, v'_2 \in I_E} \chi\left(\frac{\varpi_{v_1}}{\varpi_{v'_1}}\right) \chi^{-1}\left(\frac{\varpi_{v_2}}{\varpi_{v'_2}}\right) S_{Eis}(v_1, v'_1, v_2, v'_2).$$

Remark 3.4. *Not every cuspidal representation π' (resp. not every character ξ) has a non trivial contribution in this decomposition. Only the ones which have less conductors than $\sigma'_\chi * \varphi_0$ on every place v has. The exact choice of the base for spectral decomposition is a subtle matter. It will be described in Section 5.3. Similarly, the number of ξ 's with non zero contribution is also finite and depends on F and φ_0 .*

Lemma 3.5. *We have*

$$(3.4) \ll_{\epsilon, F, \pi} \kappa E^{\epsilon-2} Q^{(2+\kappa)\epsilon}.$$

Recall that, θ is such that no complementary series representation with parameter $> \theta$ appears as a local component of a cuspidal representation. Let $\lambda_{e, \infty}$ (resp. $\lambda_{\Phi, i\tau, \infty}$) be the eigenvalue for e (resp. $E(\Phi, i\tau)$) with respect to Δ_∞ , for e (resp. Φ) running through an orthonormal base $\mathcal{B}(\pi')$ (resp. $\mathcal{B}(\pi(\xi, \xi^{-1}))$), consisting of pure tensors of π' (resp. $\pi(\xi|\cdot|^{i\tau}, \xi|\cdot|^{-i\tau})$). For the portion (3.5)+(3.6), an adelic version of Weyl's law Theorem 2.22 is needed. From it we deduce

Lemma 3.6. *For a typical term, we have*

$$S_{cusp}(v_1, v'_1, v_2, v'_2) \ll_{\epsilon, F, \pi, \theta, \kappa, h_0} E^2 Q^{-(1/2-\theta)+\epsilon}.$$

Consequently we get

$$(3.5) \ll_{\epsilon, F, \pi, \theta, \kappa, h_0} E^2 Q^{-(1/2-\theta)+\epsilon}.$$

Lemma 3.7. *For a typical term, we have*

$$S_{Eis}(v_1, v'_1, v_2, v'_2) \ll_{\epsilon, F, \pi, \kappa, h_0} EQ^{(\kappa-1)/2+\epsilon} + E^2 Q^{-1/2+\epsilon}.$$

Consequently we get

$$(3.6) \ll_{\epsilon, F, \pi, \kappa, h_0} EQ^{(\kappa-1)/2+\epsilon} + E^2 Q^{-1/2+\epsilon}.$$

Lemmas 3.5 to 3.7 immediately imply

Lemma 3.8. *We have*

$$l^h(n(T)|\sigma'_\chi * \varphi_0|^2) \ll_{\pi, \kappa, \epsilon} E^{\epsilon-2} Q^{(2+\kappa)\epsilon} + E^2 Q^{-(1/2-\theta)+\epsilon} + EQ^{(\kappa-1)/2+\epsilon}.$$

Remark 3.9. *A comparison between the eigenvalues appearing here and those appearing in the trace of Δ_∞^{-l} should be taken into account, where $l > 1$ will be specified. We'll see this in detail later.*

Remark 3.10. *We should explain what “typical term” means in Lemmas 3.6 and 3.7. In fact, a full list of the types of positions of v_1, v'_1, v_2, v'_2 are*

Case 1: v_1, v'_1, v_2, v'_2 are distinct.

Case 2: $v_1 = v_2$ or $v'_1 = v'_2$, and there are 3 elements in $\{v_1, v'_1, v_2, v'_2\}$.

Case 3: $v_1 = v'_2$ or $v'_1 = v_2$ and there are 3 elements in $\{v_1, v'_1, v_2, v'_2\}$.

Case 4: $v_1 = v_2$ and $v'_1 = v'_2$, and there are 2 elements in $\{v_1, v'_1, v_2, v'_2\}$.

Case 5: $v_1 = v'_2$ and $v'_1 = v_2$ and there are 2 elements in $\{v_1, v'_1, v_2, v'_2\}$.

Case 6: $v_1 = v'_1 = v_2$ or $v_1 = v'_1 = v'_2$ or $v_2 = v'_2 = v_1$ or $v_2 = v'_2 = v'_1$ and there are 2 elements in $\{v_1, v'_1, v_2, v'_2\}$.

Case 7: $v_1 = v'_1 = v_2 = v'_2$.

Case 1 is dominant in the sense that there are $\simeq M_E^4$ possibilities for this case but $O(M_E^3)$ for the other cases. Therefore it is considered to be typical. We should consider each case and add together their effects to get the second assertions in Lemmas 3.6 and 3.7. But it turns out that it is **Case 1** which gives the most significant contribution in any situation that will be considered.

Now it is clear that Proposition 3.2 follows from Lemma 3.3, (3.3) and Lemma 3.8, by solving the equation

$$\min_{\kappa, E} \max(E^{\epsilon-1}, EQ^{-1/4+\theta/2}, Q^{-\kappa/2}, E^{1/2} Q^{(\kappa-1)/4+\epsilon}) = Q^{-\frac{1-2\theta}{8}+\epsilon}$$

The optimal choice is

$$E = Q^{\frac{1-2\theta}{8}}, \kappa = 1/4 + \theta/6.$$

Remark 3.11. *Intuitively, $n(T)$ translate the torus $a(\mathbb{A}^\times)$. For $F = \mathbb{Q}$ and in the classical language, the segment $\{yi : Q^{-\kappa-1} \leq y \leq Q^{\kappa-1}\}$ is translated into $\{yT + yi : Q^{-\kappa-1} \leq y \leq Q^{\kappa-1}\}$, which becomes equidistributed in the modular surface as $T \rightarrow \infty$. The T we'll choose maintain the product of local terms (3.1) near its natural least upper bound. It decreases the global term (3.2) in the sense that it kills the part of $|\varphi_0|^2$ orthogonal to the constant (one dimensional in general) subrepresentations. The amplification then takes care of the constant part of $|\varphi_0|^2$.*

Remark 3.12. *If we apply the $n(T)$ translation before the projections in (3.5) and (3.6), and use a more general result concerning the decay of matrix coefficients, then we find ourselves in the exact setting of [26], where all the technical calculations are folded in the “Ergodic Principle” in Section 2.5.3.*

4. CHOICE OF φ_0 AND LOCAL ESTIMATION

In this section we define the vector φ of Proposition 3.1. Recall that it is of the shape $\varphi = n(T)\varphi_0$. Here $\varphi_0 \in \pi$ is a pure tensor corresponding to $W_0(g) = \prod_v W_{0,v}(g_v)$ in the Kirillov model of π . Recall also that we only need to specify $W_{0,v}$ for every place $v \in V_F$.

4.1. Archimedean places. We first make the notion “Analytic Conductor” precise. The general definition, for both GL_1 and GL_2 representations, is given in 3.1.8 [26]. In this paper, we’re particularly interested in GL_1 case. Using the notations from 3.1.8 [26] and from Chapter XIV § 4 [24], one easily sees that if $F_v = \mathbb{R}$ and $\chi_v(a) = \text{sgn}(a)^m |a|^{i\varphi}$, then $\mu_{\chi_v} = \frac{i\varphi + m}{2}$, $m \in \{0, 1\}$ thus we may define

$$C(\chi_v) = 2 + \left| \frac{i\varphi + m}{2} \right|.$$

If $F_v = \mathbb{C}$ and $\chi_v(a) = \left(\frac{a}{|a|}\right)^m |a|^{i2\varphi}$, then $\mu_{\chi_v} = i\varphi + |m|/2$, we may define

$$C(\chi_v) = (2 + |i\varphi + |m|/2|)^2.$$

Lemma 4.1. *Let $\phi \in S(F_v^\times)$ (i.e. ϕ as well as all its derivatives decay faster than any polynomial of $|t^{-1}|$ as $|t| \rightarrow +\infty$ and more rapidly than any polynomial of $|t|$ as $|t| \rightarrow 0$). Let $C = C(\chi_v)$ be the analytic conductor of χ_v . Set, for $t \in F_v^\times$,*

$$G_\phi(\chi_v, t) = \int_{F_v} \phi(x) \psi_v(tx) \chi_v(x) dx.$$

Then for any $N \in \mathbb{N}$, $1/2 \leq \alpha < \beta < 1$

$$|G_\phi(\chi_v, t)| \ll_{\phi, N, \alpha, \beta} \min\left(\left(\frac{1 + |t|}{C}\right)^N, \left(\frac{C}{|t|}\right)^N, C^{1/2-\alpha} |t|^{\alpha-\beta}\right).$$

This is essentially the Lemma 3.1.14 of [26]. Let’s recall the proof: Note that C is comparable with the maximal absolute value among eigenvalues of χ_v for a fixed F_v^\times -invariant basis of differential operators of degree $[F_v : \mathbb{R}]$. The first two bounds then follow from two different kinds of integration by parts. For the third bound, apply the local functional equation as in Tate’s thesis, we obtain

$$G_\phi(\chi_v, t) = \frac{\int_{F_v} \Phi(x+t) \chi_v^{-1}(x) |x|^\alpha d^\times x}{\gamma(\chi_v, \psi_v, 1-\alpha)}$$

where $\Phi = \widehat{\phi \cdot |\cdot|^\alpha} \in S(F_v)$ is the Fourier transform of $\phi(x)|x|^\alpha$. Recall if we fix a small $\epsilon > 0$, and let $\alpha \in [1/2, 1 - \epsilon]$, by (3.5) of [26], and the third property after Theorem 3 of §3 [24]

$$|\gamma(\chi_v, \psi_v, 1 - \alpha)| \simeq_\epsilon C^{\alpha-1/2}$$

Then after some evident change of variables, one gets

$$|G_\phi(\chi_v, t)| \simeq_\epsilon C^{1/2-\alpha} |t|^\alpha \int_{F_v} \Phi(tx) |x-1|^{\alpha-1} \chi^{-1}(x-1) dx$$

But for any $\beta > 0$, $\Phi(x) \ll_{\beta, \phi} |x|^{-\beta}$, thus

$$|G_\phi(\chi_v, t)| \ll_{\epsilon, \beta, \phi} C^{1/2-\alpha} |t|^{\alpha-\beta} \int_{F_v} |x|^{-\beta} |x-1|^{\alpha-1} dx$$

The integral converges if $1/2 \leq \alpha < \beta < 1$. Under this condition, we get

$$|G_\phi(\chi_v, t)| \ll_{\alpha, \beta, \phi} C^{1/2-\alpha} |t|^{\alpha-\beta}$$

Corollary 4.2. *For any $\epsilon > 0$ there is a C_0 depending only on ϕ and ϵ , such that for $C \geq C_0$ there exists t with $|t| \in [C^{1-\epsilon}, C^{1+\epsilon}]$, s.t. $|G_\phi(\chi, t)| \geq_{\phi, \epsilon} C^{-1/2-\epsilon}$.*

Apply the Plancherel formula for $L^2(F_v)$

$$\begin{aligned} \int_{F_v} |\phi(x)|^2 dx &= \int_{F_v} |G_\phi(\chi_v, t)|^2 dt \ll_{\phi, N} \int_{|t| \leq C^{1-\epsilon}} \left(\frac{1+|t|}{C}\right)^{2N} dt + \int_{|t| \geq C^{1+\epsilon}} \left(\frac{C}{|t|}\right)^{2N} \\ &\quad + (C^{1+\epsilon} - C^{1-\epsilon}) \max_{|t| \in [C^{1-\epsilon}, C^{1+\epsilon}]} |G_\phi(\chi_v, t)|^2 \end{aligned}$$

The result follows by taking $N = 1 + \lceil \frac{1}{2\epsilon} \rceil$ ($N > 1/2 + \frac{1}{2\epsilon}$ suffices) for example.

Choose $W_{0,v} \in S(F_v^\times)$ and $T_v = t$ as in the above corollary, s.t.

$$(4.1) \quad \zeta(1/2, n(T_v)W_{0,v}, \chi_v, \psi_v) \gg_{\epsilon, W_{0,v}} C(\chi_v)^{-1/2-\epsilon}$$

Corollary 4.3. *For any $0 < \epsilon < 1/2$, and any $\sigma \in \mathbb{R}$ varying in a compact set*

$$|G_\phi(\chi_v | \cdot |_\sigma, t)| \ll_{\epsilon, \phi} \min(C^{-1/2+\epsilon}, |t|^{-1/2+\epsilon})$$

We have $|G_\phi(\chi_v, t)| \ll_{\alpha, \beta, N, \phi} \min(C^{1/2-\alpha} |t|^{\alpha-\beta}, |t|^N C^{-N}) \leq C^{-\frac{N(\beta-1/2)}{N+\beta-\alpha}}$. Take $\alpha = 1/2$, β approaching 1 and N big enough. Considering $G_\phi(\chi_v | \cdot |_\sigma, t) = G_{\phi | \cdot |_\sigma}(\chi_v, t)$ gives the result.

Remark 4.4. *If $C(\chi_v) < C_0$, note that $G_\phi(\chi_v, t)$ can be extended to an analytic function on t and χ_v not identically 0 for any fixed χ_v . Since $C(\chi_v) \leq C_0$ defines a compact region for χ_v , a routine argument gives the existence of a finite set $A \subset \mathbb{R}$ depending on ϕ and C_0 s.t. for any such χ_v , $\exists t \in A$, $|G_\phi(\chi_v, t)| \gg_{\phi, \epsilon} 1$. Thus Corollary 4.2 remains true if the condition $|t| \in [C^{1-\epsilon}, C^{1+\epsilon}]$ is replaced by $|t| \in [C/2, 2C]$. We note that $C(\chi_v) > 1$ by definition. We take $T_v = t$ accordingly.*

4.2. Non-Archimedean places. We study the local analog of the generalized Gauss sum as in the previous subsection. For simplicity, we assume that the conductor of ψ_v is \mathcal{O}_v . Take the convention $n(\varpi_v^0) = 1$.

Lemma 4.5. *Let W transform as ω_v under translations by $a(\mathcal{O}_v^\times)$. Suppose the conductor of $\omega_v \chi_v$ is $1 + \varpi_v^r \mathcal{O}_v$ and $\Re(s) = \sigma$. Then if $r > 0, l > 0$*

$$|\zeta(s + 1/2, n(\varpi_v^{-l})W, \chi_v, \psi_v)| = q^{-r/2} q^{-\sigma(l-r)} |W(\varpi_v^{l-r})|.$$

If $r > 0, l = 0$,

$$\zeta(s + 1/2, n(\varpi_v^{-l})W, \chi_v, \psi_v) = 0.$$

If $r = 0, l > 0$

$$\zeta(s + 1/2, n(\varpi_v^{-l})W, \chi_v, \psi_v) = \sum_{k=l}^{\infty} W(\varpi_v^k) \chi_v(\varpi_v)^k q_v^{-sk} - \frac{1}{q_v - 1} W(\varpi_v^{l-1}) \chi_v(\varpi_v)^{l-1} q_v^{-s(l-1)}.$$

If $r = 0, l = 0$

$$\zeta(s + 1/2, n(\varpi_v^{-l})W, \chi_v, \psi_v) = \sum_k W(\varpi_v^k) \chi_v(\varpi_v)^k q_v^{-sk}.$$

In fact for $r > 0$, $\zeta(1/2, n(\varpi_v^{-l})W, \chi_v) = W(\varpi_v^{l-r}) \chi_v(\varpi_v)^{l-r} \int_{\mathcal{O}_v^\times} \psi_v(\varpi_v^{-r} y) \omega_v \chi_v(y) d^\times y$,

the integral being a Gauss sum with absolute value $q^{-r/2}$. The following corollary is Lemma 11.7 of [31].

Corollary 4.6. *Let r be the conductor of $\omega_v \chi_v$. If π_v is spherical, take $W_{0,v}$ to be the spherical vector. If π_v is ramified, take $W_{0,v}(y) = \omega_v(y) 1_{v(y)=0}$. Then if $r > 0$*

$$|\zeta(s, n(\varpi_v^{-r})W_{0,v}, \chi_v, \psi_v)| = q_v^{-r/2}.$$

If $r = 0$,

$$\zeta(s, W_{0,v}, \chi_v, \psi_v) = L(s, \pi_v \otimes \chi_v).$$

As a consequence

$$\begin{aligned} \prod_{v < \infty} \left| \frac{L(1/2, \pi_v \otimes \chi_v)}{\zeta(1/2, n(T_v)W_{0,v}, \chi_v, \psi_v)} \right| &\leq \prod_{v < \infty, \omega_v \chi_v \text{ ramified}} \frac{1}{(1 - q_v^{-1/2+\theta})^2} C(\omega_v \chi_v)^{1/2} \\ (4.2) \quad &\ll_{\epsilon, \pi, F} \prod_{v < \infty} C(\chi_v)^{1/2+\epsilon}. \end{aligned}$$

Note that (3.1) is established by (4.1) and (4.2) once $T = (T_v)_v$ and $\varphi = n(T)\varphi_0$ are chosen, where φ_0 corresponds to $(W_{0,v})_v$.

Proposition 4.7. *The function φ_0 corresponding to $\prod_v W_{0,v}$ in the Kirillov model of π verifies $\varphi_0 \in R^s$.*

This is an obvious consequence of the discussion in the section 2.8. In fact it is easy to verify $\varphi_0 \in R_0^\infty$, then we apply Corollary 2.26.

4.3. Non-Archimedean Unitary Principal Series. We are interested in consequences of Lemma 4.5 in the case of a unitary principal series representation. We may assume $v(\psi) = 0$. For simplicity of notations, we omit the subscript v . Assume that the representation takes the form $\pi = \pi(\xi, \xi^{-1})$ for some unramified unitary character ξ of F^\times . For an integer $m \geq 0$, we are interested in vectors of π invariant by $K^0[m]$. Let W_π denote the Whittaker model of π .

Lemma 4.8. *If $W \in W_\pi$ is invariant by $K^0[m]$, then we have*

$$|W(a(y))| \ll (v(y) + 1)(m + 1)q^{m/2} \|W\| |y|^{1/2} 1_{v(y) \geq 0},$$

the implicit constant being absolute.

In fact, if we write

$$\mathcal{O}_m = \bar{\omega}^m \mathcal{O}, \mathcal{O}_n = \bar{\omega}^n \mathcal{O} - \bar{\omega}^{n+1} \mathcal{O}, 1 \leq n \leq m-1,$$

then for any $f \in \pi$, there is a sequence of complex numbers $f_n, 0 \leq n \leq m$ s.t.

$$f|_{B(\mathcal{O})wN(\mathcal{O})} = f_0, f\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) = f_n, \forall x \in \mathcal{O}_n.$$

Therefore, if

$$W(a(y)) = W_f(a(y)) = \xi^{-1}(y)|y|^{1/2} \int_F f(w_n(x))\psi(-xy)dx$$

denotes the Whittaker function of f , then we obtain, with $t = \xi(\bar{\omega})$

$$W(a(y)) = \xi^{-1}(y)|y|^{1/2} 1_{v(y) \geq 0} (f_0 - q^{-1} f_{v(y)+1} t^{2(v(y)+1)} + (1 - q^{-1}) \sum_{n=1}^{v(y)} f_n t^{2n}).$$

If $v(y) \geq m$, we rewrite

$$\begin{aligned} W(a(y)) &= \xi^{-1}(y)|y|^{1/2} 1_{v(y) \geq 0} (f_0 + (1 - q^{-1}) \sum_{n=1}^{m-1} f_n t^{2n} + \\ &\quad (\frac{t^{2m} - t^{2(v(y)+1)}}{1 - t^2} - q^{-1} \frac{t^{2m} - t^{2(v(y)+2)}}{1 - t^2}) f_m). \end{aligned}$$

By the discussion in the section 3.1.6 of [26], we have

$$\|W\|^2 = (1 - q^{-1})^{-1} (|f_0|^2 + \sum_{n=1}^{m-1} |f_n|^2 q^{-n} (1 - q^{-1}) + |f_m|^2 q^{-m}).$$

We apply Cauchy-Schwarz and get the lemma.

We apply the second case of Lemma 4.5 to the above $W = W_f$ and obtain for $\Re(s) = 1/2$

$$|l^{\cdot}|^s (n(\bar{\omega}^{-l})W) \leq (\sum_{k=l}^{\infty} |W(\bar{\omega}^k)|^2)^{1/2} (\sum_{k=l}^{\infty} q^{-k})^{1/2} + |W(\bar{\omega}^{l-1})| \frac{q^{-(l-1)/2}}{q-1}$$

$$(4.3) \quad \ll_{\epsilon} (m+1)q^{m/2} q^{-l(1-\epsilon)} \|W\|, \forall \epsilon > 0.$$

5. GLOBAL ESTIMATION

5.1. Truncation. The goal of this section is to establish Lemma 3.3.

Fix a function $h_0 \in C^\infty(\mathbb{R}^+)$ supported in $(0, 2]$ such that $h_0|_{(0,1]} = 1$ and $0 < h_0 < 1$. Denote by $\mathcal{M}(\cdot)$ the Mellin transform. For any $A > 0$, let $h_{0,A}(t) = h_0(t/A)$. The following relation is immediate:

$$|\mathcal{M}(\sigma * h_{0,Q^{-\kappa-1}})(s)| \leq 4^{|\Re(s)|} Q^{-(\kappa+1)\Re(s)} |\mathcal{M}(h_0)(s)|.$$

For any $t > 0$, choose $y_t \in \mathbb{A}^\times$ such that $|y_t| = t$, and define

$$f(t) = \int_{F^\times \setminus \mathbb{A}^{(1)}} \varphi(a(yy_t)) \chi(yy_t) d^\times y,$$

then

$$l^{\chi, \sigma * h_{0,Q^{-\kappa-1}}}(\varphi) = \int_0^{+\infty} \sigma * h_{0,Q^{-\kappa-1}}(t) f(t) d^\times t.$$

Note that $\mathcal{M}(f)(s) = l^{\chi|\cdot|^s}(\varphi)$, Mellin inversion gives,

$$\begin{aligned} |l^{\chi, \sigma * h_{0,Q^{-\kappa-1}}}(\varphi)| &= \left| \int_{\Re(s)=-1/2-\epsilon} \mathcal{M}(\sigma * h_{0,Q^{-\kappa-1}})(-s) l^{\chi|\cdot|^s}(\varphi) \frac{ds}{2\pi i} \right| \\ &\ll Q^{-(\kappa+1)(1/2+\epsilon)} \int_{\Re(s)=-1/2-\epsilon} |\mathcal{M}(h_0)(-s) l^{\chi|\cdot|^s}(\varphi)| ds. \end{aligned}$$

According to (2.6), one can write

$$\begin{aligned} l^{\chi|\cdot|^s}(\varphi) &= L(\pi \otimes \chi, s+1/2) \prod_{v|\infty} l^{\chi_v|\cdot|^s}_v(n(T_v)W_{0,v}) \prod_{v<\infty} \frac{l^{\chi_v|\cdot|^s}_v(n(T_v)W_{0,v})}{L(\pi_v \otimes \chi_v, s+1/2)} \\ &= L^{(S)}(\pi \otimes \chi, s+1/2) \prod_{v \in S} l^{\chi_v|\cdot|^s}_v(n(T_v)W_{0,v}), \end{aligned}$$

where S is the subset of places v for which $T_v \neq 0$. From Corollary 4.3 and Corollary 4.6, one sees that for each $v \in S$, $|l^{\chi_v|\cdot|^s}_v(n(T_v)W_{0,v})| \ll_{\epsilon, \varphi_0} C(\chi_v)^{-1/2+\epsilon}$ and the product of the implicit constants tends to 0 as S increases. So

$$\prod_{v \in S} l^{\chi_v|\cdot|^s}_v(n(T_v)W_{0,v}) \ll_{\epsilon, \varphi_0} Q^{-1/2+\epsilon}.$$

By the convexity bound together with bounds towards the Ramanujan-Petersson conjecture, we have

$$L^{(S)}(\pi \otimes \chi, s+1/2) \ll_\epsilon (1+|s|)^2 C(\pi \otimes \chi)^{1/2+\epsilon}, \Re(s) = -1/2 - \epsilon.$$

Note that $C(\pi \otimes \chi) \ll C(\pi)C(\chi)^2$, we finally get

$$l^{\chi, \sigma * h_{0,Q^{-\kappa-1}}}(\varphi) \ll_{\epsilon, \varphi_0, h_0} Q^{-\kappa/2+\epsilon}.$$

Similar argument, using Mellin inversion for $\Re(s) = 1/2 + \epsilon$, gives

$$l^{\chi, \sigma * (1-h_{0,Q^{\kappa-1}})}(\varphi) \ll_{\epsilon, \varphi_0, h_0} Q^{-\kappa/2+\epsilon}.$$

Lemma 3.3 is proved by taking $h = h_{0,Q^{\kappa-1}} - h_{0,Q^{-\kappa-1}}$.

We will need to exploit the Mellin transform of h further. Since for any $h \in C_c(\mathbb{R}_+)$

$$\mathcal{M}(h)(s) = (-1)^n \frac{\mathcal{M}(h^{(n)})(s+n)}{s(s+1) \cdots (s+n-1)},$$

we have, for $h = h_{0,A}$,

$$\mathcal{M}(h^{(n)})(s) = A^{s-n} \mathcal{M}(h_0^{(n)})(s).$$

For $h = h_{0,Q^{\kappa-1}} - h_{0,Q^{-\kappa-1}}$, we thus have for $n \geq 1$

$$\mathcal{M}(h)(s) = (-1)^n \frac{(Q^{(\kappa-1)s} - Q^{-(\kappa+1)s}) \mathcal{M}(h_0^{(n)})(s+n)}{s(s+1) \cdots (s+n-1)}.$$

Note that $h_0^{(n)}$ is supported in $[1, 2]$ and

$$(5.1) \quad |\mathcal{M}(h)(s)| \leq \frac{2\kappa|s| \log Q \max(Q^{(\kappa-1)\Re(s)}, Q^{-(\kappa+1)\Re(s)}) \|h_0^{(n)}\|_\infty \int_1^2 t^{\Re(s)+n} d^\times t}{|s(s+1) \cdots (s+n-1)|}$$

$$\ll_{\Re(s)+n} \frac{2\kappa \log Q \|h_0^{(n)}\|_\infty Q^{(\kappa-1)\Re(s)}}{|(s+1) \cdots (s+n-1)|}, \Re(s) \geq 0,$$

$$(5.2) \quad \ll_{\Re(s)+n} \frac{2\kappa \log Q \|h_0^{(n)}\|_\infty Q^{-(\kappa-1)\Re(s)}}{|(s+1) \cdots (s+n-1)|}, \Re(s) < 0.$$

5.2. Estimation of the Constant Contribution. Writing the Fourier expansion

$$\varphi_0(g) = \sum_{\alpha \in F^\times} W_0(a(\alpha)g),$$

we obtain

$$(a(\frac{\varpi_{v_1}}{\varpi_{v'_1}}) \varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}}) \overline{\varphi_0})_N(g) = \sum_{\alpha \in F^\times} W_0(a(\alpha)ga(\frac{\varpi_{v_1}}{\varpi_{v'_1}})) \overline{W_0(a(\alpha)ga(\frac{\varpi_{v_2}}{\varpi_{v'_2}}))}.$$

As a consequence, we get a Rankin-Selberg like equality for $\Re(s)$ big enough,

$$(5.3) \quad l^{|\cdot|^s} ((a(\frac{\varpi_{v_1}}{\varpi_{v'_1}}) \varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}}) \overline{\varphi_0})_N) = \int_{\mathbb{A}^\times} W_0(a(y)a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})) \overline{W_0(a(y)a(\frac{\varpi_{v_2}}{\varpi_{v'_2}}))} |y|^s d^\times y.$$

This integral splits into product of local factors

$$\int_{\mathbb{A}^\times} W_0(a(y)a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})) \overline{W_0(a(y)a(\frac{\varpi_{v_2}}{\varpi_{v'_2}}))} |y|^s d^\times y = \prod_{v|\infty} \int_{F_v^\times} |W_{0,v}(a(y))|^2 |y|_v^s d^\times y.$$

$$\frac{L(s+1, \pi \times \bar{\pi})}{\zeta_F(2s+2)} \prod_{v<\infty} \frac{\zeta_v(2s+2) \int_{F_v^\times} W_{0,v}(a(y)a(u_v)) \overline{W_{0,v}(a(y)a(u'_v))} |y|_v^s d^\times y}{L(s+1, \pi_v \times \bar{\pi}_v)}.$$

Here, u_v, u'_v are suitably chosen according to $\{v_1, v'_1, v_2, v'_2\}$. For almost all v , the local term equals 1. This identity admits meromorphic continuation to \mathbb{C} and is holomorphic for $\Re(s) > 0$. By the convergence of $L(s, \pi \times \bar{\pi})$, we have

$$\frac{L(s+1, \pi \times \bar{\pi})}{\zeta_F(2s+2)} \ll_{\epsilon, \pi} 1, \text{ for } \Re(s) = \epsilon > 0.$$

If v is a ramified place for π , we can always say that the corresponding local factor is bounded by some constant depending only on $\Re(s), \pi$. So we may only consider unramified places of π . On such a place, $W_{0,v}$ is spherical and is the new vector (c.f. (2.10)). If $\alpha_{1,v}, \alpha_{2,v}$ are the Satake parameters ($|\alpha_{1,v}\alpha_{2,v}| = 1$), then

$$W_{0,v}(a(\varpi_v^m)) = q_v^{-m/2} \frac{\alpha_{1,v}^{m+1} - \alpha_{2,v}^{m+1}}{\alpha_{1,v} - \alpha_{2,v}}, m \geq 0,$$

$$W_{0,v}(a(\varpi_v^m)) = 0, m < 0.$$

Hence the corresponding local term is explicitly computable. We do the calculation of one case and leave the others to the reader. Let $u_v = \varpi_v, u'_v = 1$ (i.e. $v = v_1$ and $v \neq v'_1, v_2, v'_2$), then

$$\frac{\zeta_v(2s+2) \int_{F_v^\times} W_{0,v}(a(y)a(u_v)) \overline{W_{0,v}(a(y)a(u'_v))} |y|_v^s d^\times y}{L(s+1, \pi_v \times \bar{\pi}_v)} = \frac{(tr_v - n_v \overline{tr_v} q_v^{-s-1}) q_v^{-1/2}}{1 - q_v^{-2s-2}}$$

where $tr_v = \alpha_{1,v} + \alpha_{2,v}, n_v = \alpha_{1,v} \alpha_{2,v}$. Let $\max(|\alpha_{1,v}|, |\alpha_{2,v}|) = q_v^{\theta_v}$, then $|tr_v| \ll q_v^{\theta_v}$ we get, for $\epsilon > 0$ small,

$$\left| \frac{(tr_v - n_v \overline{tr_v} q_v^{-s-1}) q_v^{-1/2}}{1 - q_v^{-2s-2}} \right| \ll q_v^{-1/2} |tr_v|, \Re(s) = \epsilon.$$

Similarly, we deal with all the other cases and get

$$\prod_{v < \infty} \frac{\zeta_v(2s+2) \int_{F_v^\times} W_{0,v}(a(y)a(u_v)) \overline{W_{0,v}(a(y)a(u'_v))} |y|_v^s d^\times y}{L(s+1, \pi_v \times \bar{\pi}_v)} \ll_{\epsilon, \pi} \prod_{v=v_1, v'_1, v_2, v'_2} q_v^{-1/2} |tr_v|.$$

Inserting it to (5.3), we obtain

$$l^{|\cdot|^s} \left(\left(a\left(\frac{\varpi_{v_1}}{\varpi_{v'_1}}\right) \varphi_0 a\left(\frac{\varpi_{v_2}}{\varpi_{v'_2}}\right) \bar{\varphi}_0 \right)_N \right) \ll_{\epsilon, \pi} \prod_{v=v_1, v'_1, v_2, v'_2} q_v^{-1/2} |tr_v|, \Re(s) = \epsilon > 0.$$

Note that

$$S_{cst}(v_1, v'_1, v_2, v'_2) = \int_{\Re(s)=\epsilon} \mathcal{M}(h)(-s) l^{|\cdot|^s} \left(\left(a\left(\frac{\varpi_{v_1}}{\varpi_{v'_1}}\right) \varphi_0 a\left(\frac{\varpi_{v_2}}{\varpi_{v'_2}}\right) \bar{\varphi}_0 \right)_N \right) \frac{ds}{2\pi i},$$

which with (5.2) gives

$$S_{cst}(v_1, v'_1, v_2, v'_2) \ll_{F, \pi, \epsilon} \kappa \log QQ^{(1+\kappa)\epsilon} E^{-2} \prod_{v=v_1, v'_1, v_2, v'_2} |tr_v|.$$

Lemma 5.1. *We have Ramanujan conjecture on average, i.e.*

$$\sum_{q_v \in I_E} |tr_v| \ll_\epsilon M_E E^\epsilon$$

In fact, by the theory of Rankin-Selberg, $L(s, \pi \times \bar{\pi})$ is meromorphic and only has possible simple poles at $s = 0, 1$. This implies $\sum_{\alpha \text{ ideal of } F, N_F(\alpha) \leq N} |\lambda_\pi(\alpha)|^2 \ll_\epsilon$

$N^{1+\epsilon}, \forall \epsilon > 0$. Here, $\lambda_\pi(\alpha)$ is the Hecke eigenvalues which coincides with tr_v when α is the prime ideal corresponding with v . Using (3.7), we obtain Lemma 3.5 from Lemma 5.1.

5.3. Estimation of the Cuspidal Contribution. The goal of this section is to establish Lemma 3.6. Recall that we are reduced to estimate

$$S_{cusp}(v_1, v'_1, v_2, v'_2) = \sum_{\pi' \text{ cuspidal}} l^h(n(T) P_{\pi'}(a(\frac{\varpi_{v_1}}{\varpi_{v'_1}}) \varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}}) \bar{\varphi}_0)).$$

The projector $P_{\pi'}$ is realized by the choice of a basis of π' , denoted by $\mathcal{B}(\pi'; v_1, v'_1, v_2, v'_2)$. It is determined by the choices of local basis of π'_v , denoted by $\mathcal{B}_v(\pi'; v_1, v'_1, v_2, v'_2)$.

When there is no confusion, we may write them shortly as \mathcal{B} resp. \mathcal{B}_v . There are related with each other by

$$\mathcal{B} = \otimes'_v \mathcal{B}_v, e \leftrightarrow (W_{e,v})_v.$$

Here, $W_{e,v}$ is the component at v of e in the Kirillov model. We may also write it as e_v if there is no confusion. According to Remark 2.18, we only need to choose \mathcal{B}_v for $v < \infty$.

Definition 5.2. Denote, for any subgroup $H \subset G(F_v)$ and $g \in G(F_v)$, $H^g = gHg^{-1}$. Then the Harish-Chandra's function $\Xi_v^{g_0}$ associated to the Borel subgroup $B(F_v)^{g_0}$ is given by, with notations in Section 2.9

$$\Xi_v^{g_0}(g) = \Xi_v(g_0^{-1}gg_0).$$

Definition 5.3. Suppose $v(\pi') = m$. For any integer n , recall that the space of $K_v^0[n]$ -invariant vectors of π'_v is of dimension $\max(n-m+1, 0)$. A **standard basis** of level n consists of, for each integer l s.t. $m \leq l \leq n$, a vector invariant by $K_v^0[l]$ and orthogonal to all the vectors invariant by $K_v^0[l-1]$, and vectors orthogonal to the space of $K_v^0[n]$ -invariant vectors. A **nice basis** of level n w.r.t. $g \in G_v$ consists of the g translates of the vectors of a standard basis of level n . Define **the maximal compact subgroup** K_v^* of G_v **associated with the above nice basis** to be

$$K_v^* = K_v^g.$$

If \mathcal{B}_v is a standard or nice basis of level n , we write \mathcal{B}_v^* to be the elements in \mathcal{B}_v invariant by $K_v^0[n]$ or its corresponding translate. We also call the basis as in Remark 2.18 standard. At an infinite place, we define $\mathcal{B}_v^* = \mathcal{B}_v$. We write

$$\mathcal{B}^* = \otimes'_v \mathcal{B}_v^*$$

If \mathcal{B}_v is of level n , then

$$(5.4) \quad \sum_{e_v \in \mathcal{B}_v^*} \dim(K_v e_v) \ll q_v^n$$

We choose \mathcal{B}_v and K_v^* explicitly as follows:

Case 1: v is different from v_1, v'_1, v_2, v'_2 resp. $v = v_1 = v_2$ resp. $v = v'_1 = v'_2$. Since $a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0$ is $K_v^0[v(\varphi_0)]$ resp. $K_v^0[v(\varphi_0)]^{a(\varpi_v)}$ resp. $K_v^0[v(\varphi_0)]^{a(\varpi_v^{-1})}$

invariant, we take \mathcal{B}_v to be a standard basis of level $v(\varphi_0)$ resp. a nice basis of level $v(\varphi_0)$ w.r.t. $a(\varpi_v)$ resp. a nice basis of level $v(\varphi_0)$ w.r.t. $a(\varpi_v^{-1})$.

Case 2: $v = v_1$ or v_2 resp. $v = v'_1$ or v'_2 . Since $a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0$ is $K_v^0[v(\varphi_0) + 1]^{a(\varpi_v)}$ resp. $K_v^0[v(\varphi_0) + 1]$ invariant, we take \mathcal{B}_v to be a nice basis of level $v(\varphi_0) + 1$ w.r.t. $a(\varpi_v)$ resp. a standard basis of level $v(\varphi_0) + 1$.

Case 3: $v = v_1 = v'_2$ or $v = v_2 = v'_1$. Since $a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0$ is $K_v^0[v(\varphi_0) + 2]^{a(\varpi_v)}$ invariant, we take \mathcal{B}_v to be a nice basis of level $v(\varphi_0) + 2$ w.r.t. $a(\varpi_v)$.

Then we rewrite

$$(5.5) \quad S_{cusp}(v_1, v'_1, v_2, v'_2) = \sum_{\pi'} \sum_{e \in \mathcal{B}^*} \langle a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0, e \rangle l^h(n(T)e).$$

We have

$$(5.6) \quad l^h(n(T)e) = \int_{\Re(s)=0} \mathcal{M}(h)(-s) l^{|\cdot|^s} (n(T)e) \frac{ds}{2\pi i},$$

and since the vector e is a pure tensor,

$$l^{|\cdot|^s}(n(T)e) = L(s+1/2, \pi') \prod_{v|\infty} l^{|\cdot|^s}(n(T_v)W_{e,v}) \prod_{v<\infty} \frac{l^{|\cdot|^s}(n(T_v)W_{e,v})}{L(s+1/2, \pi'_v)}.$$

The convexity bound gives

$$|L(s+1/2, \pi')| \ll_{\epsilon} C(\pi' \otimes |\cdot|^s)^{1/4+\epsilon} \ll (1+|s|)^{1/2+\epsilon} C(\pi')^{1/4+\epsilon}, \forall \epsilon > 0.$$

Lemma 5.4. *Let I_1 be the set of places v s.t. $v \in \{v_1, v'_1, v_2, v'_2\}$ and π_v is unramified, π'_v is ramified. Let I_2 be the set of places v s.t. $v \in \{v_1, v'_1, v_2, v'_2\}$ and π_v, π'_v are unramified. Then we have $\forall \epsilon > 0$*

$$|l^{|\cdot|^s}(n(T)e)| \leq_{\epsilon, \theta, \varphi_0} (1+|s|)^{1/2+\epsilon} |T|^{-1/2+\theta+\epsilon} \lambda_{e, \infty}^{3/4+\epsilon} \prod_{v \in I_1} q_v^{v(\pi')/4+\epsilon} \prod_{v \in I_2} \dim(K_v^* e_v)^{1/2}.$$

Write

$$M_v(e) = \sup_{s \in i\mathbb{R}} |C(\pi'_v)^{1/4+\epsilon} l^{|\cdot|^s}(n(T_v)W_{e,v})|, \forall v|\infty,$$

$$M_v(e) = \sup_{s \in i\mathbb{R}} |C(\pi'_v)^{1/4+\epsilon} \frac{l^{|\cdot|^s}(n(T_v)W_{e,v})}{L(s+1/2, \pi'_v)}|, \forall v < \infty.$$

To proof Lemma 5.4, we estimate the local terms $M_v(e)$ case by case. This is technical and will be given in the following subsections. Lemma 5.4 will be a consequence of Corollary 5.6, 5.9, Lemma 5.7, 5.10, 5.11, as well as Lemma 2.8 and the remark following it (with $\|e\|_{X(F)} = 1$). Recall the following bound resulting from (5.1)

$$\int_{i\mathbb{R}} |\mathcal{M}(h)(-s)| (1+|s|)^{1/2+\epsilon} \frac{ds}{2\pi i} \ll_{\epsilon} 2\kappa \log Q \|h_0^{(3)}\|_{\infty}.$$

From it we obtain

$$\begin{aligned} (5.5) &\leq \sum_{\pi'} \sum_e |\langle a(\frac{\varpi_{v_1}}{\varpi_{v'_1}}) \varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}}) \overline{\varphi}_0, e \rangle| \times |l^h(n(T)e)| \\ &\ll_{F, \epsilon, \theta, \varphi_0, h_0} |T|^{-1/2+\theta+\epsilon} \|P_{\text{cusp}}(\Delta_{\infty}^2 a(\frac{\varpi_{v_1}}{\varpi_{v'_1}}) \varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}}) \overline{\varphi}_0)\| \times \\ (5.7) &\sqrt{\sum_{\pi'} \sum_{e_{\infty} \in \otimes_{v|\infty} \mathcal{B}_v^*} \lambda_{e, \infty}^{-5/2+\epsilon} \prod_{v<\infty, T_v \neq 0} (v(\varphi_0) + 1) \prod_{v<\infty, T_v=0, v(\varphi_0)>0} (v(\varphi_0) + 3) S(I_1, I_2)} \end{aligned}$$

with

$$S(I_1, I_2) = \prod_{v \in I_1} \sum_{e_v \in \mathcal{B}_v^*} q_v^{v(\pi')/2+\epsilon} \prod_{v \in I_2} \sum_{e_v \in \mathcal{B}_v^*} \dim(K_v^* e_v).$$

It is easy to see that $\|P_{\text{cusp}}(\Delta_{\infty}^2 a(\frac{\varpi_{v_1}}{\varpi_{v'_1}}) \varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}}) \overline{\varphi}_0)\|$ is bounded above by some Sobolev norm of φ_0 . In a typical situation as mentioned in Remark 3.10, distinguishing $v(\pi') = 0$ and $v(\pi') = 1$ and using (5.4) we get

$$S(I_1, I_2) \ll \prod_{v \in \{v_1, v'_1, v_2, v'_2\}} q_v \ll E^4.$$

Therefore

$$(5.7) \ll_{\varphi_0} E^2 (\text{trace of } \Delta_{\infty}^{-5/2+\epsilon})^{1/2},$$

which proves the first part of Lemma 3.6. The second part follows from similar bounds for $S(I_1, I_2)$ in the other 6 situations mentioned in Remark 3.10.

5.3.1. **At v such that $T_v \neq 0$.** In this case, \mathcal{B}_v is given by the first case of **Case 1**, hence is standard. Note that

$$|l^{|\cdot|^s}(n(T_v)W_{e,v})|^2 = \int_{F_v^\times} \langle n(-T_v)a(y)n(T_v)W_{e,v}, W_{e,v} \rangle |y|^s d^\times y.$$

By Theorem 2.31, we get,

(5.8)

$$|l^{|\cdot|^s}(n(T_v)W_{e,v})|^2 \leq A_v(\epsilon) \dim(K_v e_v) \|W_{e,v}\|^2 \int_{F_v^\times} \Xi_v(n(-T_v)a(y)n(T_v))^{1-2\theta-\epsilon} d^\times y.$$

Lemma 5.5. *For any $\epsilon > 0$, we have*

$$|l^{|\cdot|^s}(n(T_v)W_{e,v})| \ll_{\epsilon, \theta} |T_v|_v^{-1/2+\theta+\epsilon} \dim(K_v e_v)^{1/2} \|W_{e,v}\|.$$

Corollary 5.6. *There exist a constant $C(\theta, \epsilon)$ depending only on θ and ϵ s.t.*

If $v|\infty$, then we have

$$M_v(e) \leq C(\theta, \epsilon) \lambda_{e,v}^{3/4+\epsilon} |T_v|_v^{-1/2+\theta+\epsilon} \|W_{e,v}\|.$$

If $v < \infty$, then

$$M_v(e) \leq C(\theta, \epsilon) |T_v|_v^{-1/2+\theta+\epsilon} q_v^{3v(\varphi_0)/4} \|W_{e,v}\|.$$

Note that $\dim(K_v e_v), C(\pi'_v) \ll \lambda_{e,v}$ if $v|\infty$, and e_v is $K_v^0[v(\varphi_0)]$ invariant by the choice of \mathcal{B}_v^* , $v(\pi') \leq v(\varphi_0)$ if $v < \infty$, we deduce the corollary from the lemma by noting that

$$[K_v : K_v^0[v(\varphi_0)]] \ll q_v^{v(\varphi_0)}.$$

Let us now prove Lemma 5.5 place by place.

At a Real Place : $F_v = \mathbb{R}$

Recall the (bi- K_v -invariant, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ -invariant) Harish-Chandra's function as in [15] 5.2.2 is given by:

$$\Xi_v\left(\begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix}\right) = \mathfrak{P}_{-1/2}(\cosh r), r > 0.$$

For some absolute constants $\alpha, \beta > 0$, we have

$$\mathfrak{P}_{-1/2}(\cosh r) \leq e^{-r/2}(\alpha + \beta r).$$

We make a change of variable $t = \frac{y + y^{-1}}{2}$ and get

$$\begin{aligned} & \int_{\mathbb{R}^\times} \Xi_v(n(-T_v)a(y)n(T_v))^{1-2\theta} d^\times y \\ & \leq 2(1 + T_v^2)^{-\frac{1-2\theta}{2}} (1 + \log(1 + T_v^2))^{1-2\theta} \int_1^\infty (t-1)^{-1/2+\theta} (\alpha' + \beta \log t)^{1-2\theta} \\ & \quad + t^{-1/2+\theta} (\alpha' + \beta \log(t+1))^{1-2\theta} \frac{dt}{\sqrt{t^2-1}} \\ & \ll_\theta (1 + T_v^2)^{-\frac{1-2\theta}{2}} (1 + \log(1 + T_v^2))^{1-2\theta}. \end{aligned}$$

We get the lemma at v using (5.8).

At a Complex Place : $F_v = \mathbb{C}$

The Harish-Chandra's function as in [15] 5.2.1 is given by:

$$\Xi_v\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) = \frac{2 \log t}{t - t^{-1}}, t > 0.$$

When we evaluate it at $n(-T_v)a(y)n(T_v)$, the corresponding t satisfies

$$t^2 + t^{-2} = |y| + |y|^{-1} + \frac{|T_v|^2 |y - 1|^2}{|y|}.$$

This expression being invariant by the change of variable $y \mapsto y^{-1}$, we get, with the change of variable $r = \frac{|y| + |y|^{-1}}{2}$

$$\begin{aligned} \int_{\mathbb{C}^\times} \Xi_v(n(-T_v)a(y)n(T_v))^{1-2\theta} d^\times y &= 2 \int_{|y|>1} \left(\frac{2 \log t}{t - t^{-1}}\right)^{1-2\theta} d^\times y \\ &\leq 2(2(1 + |T_v|^2))^{-\frac{1-2\theta}{2}} (\log 2(1 + |T_v|^2))^{1-2\theta} \cdot 2\pi \int_1^\infty \left(\frac{1 + \frac{\log(r+1)}{\log 2}}{\sqrt{r-1}}\right)^{1-2\theta} \frac{dr}{\sqrt{r^2-1}} \\ &\ll_\theta (1 + T_v^2)^{-\frac{1-2\theta}{2}} (1 + \log(1 + T_v^2))^{1-2\theta}. \end{aligned}$$

We get the lemma at v using (5.8).

At a Non Archimedean Place

The values of the Harish-Chandra function associated with the standard Borel subgroup can be inferred from the Macdonald formula, Theorem 4.6.6 of [2], by letting $\alpha_1 \rightarrow 1, \alpha_2 = 1$.

$$\Xi_v(n) = \Xi_v\left(\begin{pmatrix} \varpi_v^n & 0 \\ 0 & 1 \end{pmatrix}\right) = q_v^{-n/2} + n q_v^{-n/2} \frac{1 - q_v^{-1}}{1 + q_v^{-1}}, n \geq 0$$

Apply (42) of [15] to the torus $\mathbb{T} = n(-T_v)Z_v A_v n(T_v)$, the local integral can be calculated and bounded as, with $d = \max(0, -v(T_v))$

$$\begin{aligned} &q_v^{d_v/2} \int_{F_v^\times} \Xi_v(n(-T_v)a(y)n(T_v))^{1-2\theta} d^\times y \\ &= 2 \sum_{n>2d} \Xi_v(n)^{1-2\theta} + \sum_{n=1}^{d-1} \frac{q_v^{d-n} - q_v^{d-n-1}}{q_v^d - q_v^{d-1}} \Xi_v(2(d-n))^{1-2\theta} \\ &\quad + \frac{1}{q_v^d - q_v^{d-1}} \Xi_v(0)^{1-2\theta} + \frac{q_v^d - 2q_v^{d-1}}{q_v^d - q_v^{d-1}} \Xi_v(2d)^{1-2\theta} \\ &\ll C(\theta) \max(1, |T_v|)^{-(1-2\theta)} (1 + \max(1, \log |T_v|))^{2-2\theta}. \end{aligned}$$

We get the lemma at v using (5.8) and conclude the lemma. We record the following estimation: for some constant $C'(\theta)$ depending only on θ ,

$$(5.9) \quad q_v^{d_v/2} \int_{F_v^\times} \Xi_v(a(y))^{1-2\theta} d^\times y \leq 2 \sum_{n>0} (n+1) q_v^{-n(1/2-\theta)} + 1 \leq C'(\theta)$$

5.3.2. At v such that $T_v = 0$, π_v ramified. The number of such places is finite and depends only on π . \mathcal{B}_v is given by **Cases 1,2,3**. Applying (5.8) (with $T_v = 0$) and (5.9) we get similarly to section 5.3.1

Lemma 5.7. *For $\forall \epsilon > 0$, there is a constant $C(\theta, \epsilon)$ s.t.*

$$M_v(e) \leq C(\theta, \epsilon) q_v^{3v(\varphi_0)/4+3/2+\epsilon} \|W_{e,v}\|.$$

This follows from the bound $\dim(K_v^* e_v) \leq q_v^{v(\varphi_0)+2}$: a suitable translate of e_v is at most $K_v^0[v(\varphi_0) + 2]$ invariant, and $v(\pi') \leq v(\varphi_0) + 2$.

5.3.3. At v such that $T_v = 0$, π_v unramified, π'_v ramified. In that case $v \in \{v_1, v'_1, v_2, v'_2\}$. So the number of possible places is at most 4 and $v(\pi') \leq 2$. By the theory of new vectors and conductor, we know that $y \mapsto W_{e,v}(a(y))$ is supported in $\{y \in F_v^\times : v(y) = m\}$ for some integer $m \geq 0$ (c.f. [10]). Therefore, by Cauchy-Schwarz (since $s \in i\mathbb{R}$) we deduce

Lemma 5.8. *We have*

$$\left| \frac{l^{|\cdot|^s}(W_{e,v})}{L(s+1/2, \pi'_v)} \right| = |l^{|\cdot|^s}(W_{e,v})| \leq q_v^{-d_v/2} \|W_{e,v}\|.$$

Corollary 5.9. *For any $\epsilon > 0$, there is $C(\epsilon)$ depending only on ϵ s.t.*

$$M_v(e) \leq C(\epsilon) q_v^{v(\pi')/4+\epsilon} q_v^{-d_v/2} \|W_{e,v}\|.$$

5.3.4. At v such that $T_v = 0$, π_v unramified, π'_v unramified. If $v \notin \{v_1, v'_1, v_2, v'_2\}$ then e_v is spherical and we have

Lemma 5.10. *For $v \notin \{v_1, v'_1, v_2, v'_2\}$, we have*

$$M_v(e) = \frac{\|W_{e,v}\|}{\sqrt{L(1/2, \pi'_v \times \pi'_v)}}.$$

Note that almost all v are in this category.

If $v \in \{v_1, v'_1, v_2, v'_2\}$, then we apply (5.8) and (5.9) to get

Lemma 5.11. *For $v \in \{v_1, v'_1, v_2, v'_2\}$, there is a constant $C(\theta, \epsilon)$ depending only on θ and ϵ ,*

$$M_v(e) \leq C(\theta, \epsilon) \dim(K_v^* e_v)^{1/2} \|W_{e,v}\|$$

5.4. Estimation of the Eisenstein Contribution. The goal of this section is to establish Lemma 3.7. First we rewrite

$$S_{Eis}(v_1, v'_1, v_2, v'_2) = \sum_{\xi \in F^\times \setminus \widehat{\mathbb{A}}(1)} \sum_{\Phi \in \mathcal{B}(\pi(\xi, \xi^{-1}))} \int_{-\infty}^{\infty} \langle a(\frac{\varpi_{v_1}}{\varpi_{v'_1}}) \varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}}) \overline{\varphi}_0, E(\Phi, i\tau) \rangle \cdot l^h(n(T)(E(\Phi, i\tau) - E_N(\Phi, i\tau))) d\tau.$$

The treatment of $l^h(n(T)(E(\Phi, i\tau) - E_N(\Phi, i\tau)))$ is similar to that of $l^h(n(T)e)$ in the previous section, except that we can take $\theta = 0$. One starts with

$$l^h(n(T)(E(\Phi, i\tau) - E_N(\Phi, i\tau))) = \int_{\Re(s) \gg 1} \mathcal{M}(h)(-s) l^{|\cdot|^s}(n(T)(E(\Phi, i\tau) - E_N(\Phi, i\tau))) \frac{ds}{2\pi i}$$

with

$$(5.10) \quad |l|^s (n(T)(E(\Phi, i\tau) - E_N(\Phi, i\tau))) = \Lambda(s + i\tau + 1/2, \xi) \Lambda(s - i\tau + 1/2, \xi^{-1}) \\ \cdot \prod_v \frac{|l|^s (n(T_v)W_{\Phi_{i\tau}, v})}{L(s + i\tau + 1/2, \xi_v) L(s - i\tau + 1/2, \xi_v^{-1})}$$

where $\Lambda(\cdot, \xi)$ is the completed (GL_1) L -function. (5.10) has an analytic continuation and admits simple poles at $s = 1/2 \pm i\tau$ only when $\xi = 1$ is the trivial character and $\tau \neq 0$. We proceed by shifting the contour to $\Re s = 0$. We need to bound the contribution of the poles. The local factors for which $T_v \neq 0$ are bounded by using Corollary 4.3, (4.3). For those for which $v \in \{v_1, v'_1, v_2, v'_2\}$, $T_v = 0$, we use instead

$$|l|^{|1/2 \pm i\tau|} (W_{\Phi_{i\tau}, v}) \leq \|W_{\Phi_{i\tau}, v}\| \int_{\text{supp } W_{\Phi_{i\tau}, v}} |y| d^\times y.$$

Together with (5.2), we deduce that they are of size $O_\epsilon(Q^{(\kappa-1)/2+\epsilon} E)$.

In order to bound the contribution on the line $\Re(s) = 0$, we use the bound

$$|l|^s (n(T_v)W_{\Phi_{i\tau}, v})|^2 \leq \dim(K_v W_{\Phi_{i\tau}, v}) \|W_{\Phi_{i\tau}, v}\|^2 \int_{F_v^\times} \Xi_v(n(-T_v)a(y)n(T_v)) d^\times y.$$

Since Ξ_v is a matrix coefficient, one always has $\Xi_v \leq 1$, so $\Xi_v \leq \Xi_v^{1-\epsilon}$ for any $\epsilon > 0$. We get

$$|l|^s (n(T_v)W_{\Phi_{i\tau}, v})| \ll_\epsilon (1 + |T_v|)^{-1/2+\epsilon} (\dim(K_v W_{\Phi_{i\tau}, v}))^{1/2} \|W_{\Phi_{i\tau}, v}\|.$$

Similarly to the previous section, using the convexity bounds for GL_1 L -functions, the contribution on the line $\Re(s) = 0$ is bounded by $O_{\epsilon, F, \pi, \kappa, h_0}(E^2 Q^{-1/2+\epsilon})$. This completes the proof of Lemma 3.7.

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